DETERMINANTAL REPRESENTATION OF THE MOORE–PENROSE INVERSE MATRIX OVER THE QUATERNION SKEW FIELD

I. I. Kyrchei

Within the framework of the theory of column and row determinants, we have obtained the determinantal representation of the Moore–Penrose inverse matrix over the quaternion skew field.

Introduction

The existence, uniqueness, and complete-rank representation of the Moore–Penrose inverse matrix over the quaternion skew field $\mathbb{H}$ as well as its application for the solution of matrix quaternion equations were considered, in particular, in [7, 12–16, 20, 22]. The use of the complex representation of quaternion algebra is the key moment in these works. At the same time, the problem of determinantal representation of the Moore–Penrose matrix over the quaternion skew field $\mathbb{H}$ has not been solved up to now. Moreover, within the framework of any theory of determinants over the quaternion skew field known at present, researchers have not succeeded in obtaining the determinantal representation of inverse matrix. The problem lies in the fact that, as shown in [4], there is no determinantal functional over a skew field that preserves all properties characteristic of it in the complex case. For example, the Dieudonné and Stady determinants take their value not in the skew field itself but in the field, i.e., at its center. In addition, they do not have the property of expansion in any column or row of the matrix. The Moore determinant is defined in terms of permutations exclusively for Hermitian matrices. Another determinant in terms of permutations, i.e., the Chen determinant [9], does not satisfy the key property of determinant: its singularity for noninvertible matrices. The double determinant, also introduced by Chen, does not have the property of expansion in any column or row of the matrix.

The determinantal representation of inverse matrix over the quaternion skew field was constructed in [1] within the framework of the theory of column and row determinants. In the present work, within the framework of the theory of new matrix functionals over the quaternion skew field and column and row determinants introduced in [1], we have obtained the determinantal representation of Moore–Penrose inverse matrix over the quaternion skew field. In Sec. 1, we consider some known facts from the theory of eigenvalues of a quaternion matrix and its singular decomposition as well as introduce the notion of characteristic polynomial for an Hermitian matrix and study its coefficients. We also introduce the notion of Moore–Penrose inverse matrix over the quaternion skew field and its limiting representation. In Sec. 2, we prove the theorem on the determinantal representation of Moore–Penrose inverse matrix for an arbitrary matrix $A \in \mathbb{H}^{m \times n}$ over the quaternion skew field.

We use the following notation: We denote by $\mathcal{M}(n, \mathbb{H})$ a ring of square matrices of $n$th order, by $\mathbb{H}^{m \times n}$ the set of all $m \times n$ matrices over the quaternion skew field $\mathbb{H}$, and by $\mathbb{H}_r^{m \times n}$ its subset of matrices of rank $r$. We here interpret the rank of a matrix as its column rank, i.e., the maximal quantity of right-linearly independent columns of the matrix, or its row rank equal to the column one, i.e., the maximal quantity of left-linearly independent rows of the matrix. Let $A_{ij}$ be the submatrix of a matrix $A \in \mathcal{M}(n, \mathbb{H})$ that can be ob-
tained by deleting the \(i\)th row and \(j\)th column. We denote the \(j\)th column of the matrix \(A\) by \(a_{ij}\) and its \(i\)th row by \(a_{ij}\). Finally, suppose that \(A_{ij}(b)\) is the matrix that can be obtained from \(A\) by replacing its \(j\)th column by the column \(b\), and \(A_{ij}(b)\) is the matrix obtained from \(A\) by replacing its \(i\)th row by the row \(b\).

1. Singular Decomposition of Quaternion Matrices and Moore–Penrose Inverse Matrix

In view of noncommutativity over skew field, it is customary to distinguish the left and right eigenvalues of a quaternion matrix \(A \in M(n, \mathbb{H})\), i.e., the solutions \(\lambda\) of equations \(A \cdot x = \lambda \cdot x\) and \(A \cdot x = x \cdot \lambda\). The development of the theory of left eigenvalues is traced in [11, 17, 19]. The theory of right eigenvalues of quaternion matrices is more developed. In particular, we should mention here papers [5, 21]. We present two propositions from the theory of right eigenvalues.

**Proposition 1** [21]. Let \(A \in M(n, \mathbb{H})\) be an Hermitian matrix. Then the right eigenvalues of the matrix \(A\) are real, and their quantity is equal to \(n\).

**Definition 1.** Let \(U \in M(n, \mathbb{H})\). If \(U^*U = UU^* = I\), where \(I\) is the unit matrix, then \(U\) is called a unitary matrix.

**Proposition 2** [21]. A matrix \(A \in M(n, \mathbb{H})\) is Hermitian if and only if there exists a unitary matrix \(U \in M(n, \mathbb{H})\) such that \(A = UDU^*\), where \(D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\) and \(\lambda_i \in \mathbb{R}\) are eigenvalues of the matrix \(A\) for all \(i = 1, \ldots, n\).

Let \(A \in M(n, \mathbb{H})\) be an Hermitian matrix. Then, for its arbitrary right eigenvalue \(\lambda \in \mathbb{R}\), we have \(A \cdot x = x \cdot \lambda = \lambda \cdot x\). This means that all right eigenvalues of the Hermitian matrix are also its left eigenvalues. Then, for real left eigenvalues \(\lambda \in \mathbb{R}\), the matrix \(\lambda I - A\) is also Hermitian. Provided that \(t \in \mathbb{R}\), we introduce for the Hermitian matrix \(A \in M(n, \mathbb{H})\) the notion of its characteristic polynomial, taking \(p(t) := \det(tI - A)\). Here and below, the determinant of an Hermitian matrix is defined according to [1, Remark 3.1]. The roots of the characteristic polynomial of an Hermitian matrix are its real left eigenvalues, which are simultaneously its right eigenvalues. We now investigate, by analogy with the commutative case (see, e.g., [2]), coefficients of the characteristic polynomial. First, we present the following auxiliary proposition:

**Lemma 1.** Suppose that \(A \in M(n, \mathbb{H})\) is an Hermitian matrix, and its columns \(i_1, \ldots, i_k\) coincide with the unit vectors \(e_{i_1}, \ldots, e_{i_k}\). Then \(\det A\) is equal to the principal minor of the matrix \(A\) that can be obtained by deleting the rows and columns \(i_1, \ldots, i_k\).

**Proof.** Note that if \(A \in M(n, \mathbb{H})\) is an Hermitian matrix whose columns \(i_1, \ldots, i_k\) coincide with the unit column vectors \(e_{i_1}, \ldots, e_{i_k}\), then its rows coincide with the unit row vectors \(e_{i_1}, \ldots, e_{i_k}\). Since \(\det A = \text{cdet}_{i_1} A\) [1, Theorem 3.1], we expand the determinant \(\det A\) in the column \(i_1\) [1, Lemma 2.2], where \(a_{i_1 j} = 0\) for all \(k \neq i_1\) and \(a_{i_1 i_1} = 1\). Then

\[
\det A = -\text{cdet}_{i_1} A_{i_1}^{11}(a_{11}) \cdot a_{i_1 i_1} + \ldots + \text{cdet}_{i_1} A_{i_1}^{i_1 i_1} \cdot a_{i_1 i_1} + \ldots - \text{cdet}_{i_1} A_{i_1}^{i_1 i_i} \cdot a_{i_1 i_1} = -\text{cdet}_{i_1} A_{i_1}^{11}(a_{11}) \cdot 0 + \ldots + \text{cdet}_{i_1} A_{i_1}^{i_1 i_1} \cdot 1 + \ldots - \text{cdet}_{i_1} A_{i_1}^{i_1 i_i} \cdot 0 = \text{cdet}_{i_1} A_{i_1}^{i_1 i_i}.
\]