We consider different methods for constructing a chain of embedded spaces of first order splines (not necessarily smooth and polynomial), based on a local enlargement of an irregular grid. We present the corresponding wavelet decompositions and prove that the decomposition operators are commutative relative to the order of removing grid points.

Bibliography: 6 titles.

Embedding of spaces of smooth polynomial splines and the corresponding wavelet decompositions on infinite embedded grids were studied in many works (cf., for example, [1]–[3] and the references therein). In particular, smooth nonpolynomial splines were considered in [4]–[6]. A grid can be enlarged by removing grid points one at a time. Hence the question arises, whether the corresponding decomposition operation is commutative. Such considerations are based on approximate relations, owing to which it is possible to obtain wavelet decompositions of spline spaces with different smoothness and such that their approximate properties are asymptotically optimal with respect to the $N$-diameter of standard compact sets. The original numerical flow is regarded as a sequence of coefficients of decomposition with respect to the coordinate splines in the space constructed on the original (small) grid. This space is projected onto the embedded spline space (on an enlarged grid). As a result, we get a grid obtained by splitting the original numerical information flow into the basic flow (formed by the coefficients of decomposition relative to the coordinate splines of the embedded space) and the wavelet numerical flow which can be used to restore the original numerical flow. Nonsmooth splines are simpler than smooth ones; wavelet decompositions of a space of such splines on a segment (with a finite grid) are good for computations purposes. In this paper, based on a local enlargement of an irregular grid, we present a simple method for constructing chains of embedded spline spaces on a segment.
1 Preliminaries

We recall some notions and notation (cf., for example, [5]) which will be used below. We introduce an infinite grid on an interval \((\alpha, \beta) \subset \mathbb{R}^1, X=\{x_j\}_{j \in \mathbb{Z}},\)

\[
X : \ldots < x_{-1} < x_0 < x_1 < \ldots, \\
\alpha = \lim_{j \to -\infty} x_j, \quad \beta = \lim_{j \to +\infty} x_j \quad \forall j \in \mathbb{Z}.
\]

We set

\[
S_j = (x_j, x_{j+1}) \cup (x_{j+1}, x_{j+2}), \quad G = \bigcup_{j \in \mathbb{Z}} (x_j, x_{j+1}).
\]

We consider an ordered set of column-vectors \(A = \{a_j\}_{j \in \mathbb{Z}}, a_j \in \mathbb{R}^2,\) such that for any \(j \in \mathbb{Z}\) the matrices \(A_j = (a_j-1, a_j)\) are not singular. Such a set is called a complete chain of vectors. The set of all complete chains is denoted by \(\mathcal{A}\).

For a two-component vector-valued function \(\varphi(t), t \in G,\) with linearly independent components on any interval \((a, b) \subset G,\) we introduce functions \(\omega_j(t), t \in G,\) by the approximate relations

\[
a_{i-1} \omega_{i-1}(t) + a_i \omega_i(t) \equiv \varphi(t) \quad \forall t \in (x_i, x_{i+1}) \quad \forall i \in \mathbb{Z},
\]

\[
\omega_j(t) \equiv 0 \quad \forall t \in G \setminus S_j \quad \forall j \in \mathbb{Z}.
\]

Thus, \(\omega_j(t)\) are defined on \(G\) by the formula

\[
\omega_j(t) = \begin{cases}
\frac{\det(a_{j-1}, \varphi(t))}{\det(a_{j-1}, a_j)}, & t \in (x_j, x_{j+1}), \\
\frac{\det(\varphi(t), a_{j+1})}{\det(a_j, a_{j+1})}, & t \in (x_{j+1}, x_{j+2}), \\
0, & t \in G \setminus S_j.
\end{cases}
\]

We set

\[
\mathcal{S} = \mathcal{S}(X, A, \varphi) = Cl_p \mathcal{L} \{\omega_j\}_{j \in \mathbb{Z}},
\]

where \(\mathcal{L} \{\ldots\}\) denotes the linear hull of the elements in the curly brackets and \(Cl_p\) means the closure in the topology of pointwise convergence. A function in \(\mathcal{S}\) is called an \((X, A, \varphi)\)-spline.

2 Auxiliaries

Following [5], we recall some definitions and assertions. We consider the linear space \(C(c, d)\) of functions \(u(t)\) in \(C(c, d)\) such that \(\lim_{t \to c+0} u(t)\) and \(\lim_{t \to d-0} u(t)\) exist and are finite. We also introduce the spaces

\[
C_X = \bigotimes_{i \in \mathbb{Z}} C(x_i, x_{i+1}), \quad C_X^\infty = \{u \mid u^{(i)} \in C_X \forall i = 0, 1, \ldots, S\}.
\]

\footnote{In what follows, we consider infinite series of the form \(\sum_j c_j \omega_j^t, c_j \in \mathbb{R}^1,\) where the sum is taken over all integers \(j \in \mathbb{Z}.\) We note that for each fixed \(t \in (\alpha, \beta)\) such a series contains finitely many nonzero terms. Therefore, for any sequence \(\{c_j\}_{j \in \mathbb{Z}}\) the coefficients \(c_j \in \mathbb{R}^1\) of the series converge (in the sense of the pointwise convergences).}