HARMONIC ANALYSIS ON THE INFINITE-DIMENSIONAL UNITARY GROUP

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The goal of harmonic analysis on the infinite-dimensional unitary group is to decompose a certain family of unitary representations of this group, which is a substitute for the nonexistent regular representation and depends on two complex parameters (Olshanski, 2003). In the case of noninteger parameters, the decomposing measure is described in terms of determinantal point processes (Borodin and Olshanski, 2005). The aim of the present paper is to describe the decomposition for integer parameters; in this case, the spectrum of the decomposition changes drastically. A similar result was earlier obtained for the infinite symmetric group (Kerov, Olshanski, and Vershik, 2004), but the case of the unitary group turned out to be much more complicated. In the proof we use Gustafson’s multilateral summation formula for hypergeometric series. Bibliography: 6 titles.

INTRODUCTION

Elements of the group $U(\infty)$ are infinite unitary matrices $U = [U_{ij}], i, j = 1, 2, \ldots$, such that $U_{ij} = \delta_{ij}$ for sufficiently large $i + j$. Along with $S(\infty)$, this is an example of a “large” group. According to an idea developed in [5, 4], the main goal of harmonic analysis on such a group is to decompose the most “natural” unitary representations of this group into irreducible components.

Let $G = U(\infty) \times U(\infty)$, and let $K$ be the diagonal subgroup in $G$ isomorphic to $U(\infty)$. Then $(G, K)$ is a Gelfand pair. Olshanski [5] constructed a family $T_{zw}$ of representations of $G$ indexed by two complex parameters $z$ and $w$ satisfying the condition $\text{Re}(z + w) > -\frac{1}{2}$. For any such $z$ and $w$, the representation $T_{zw}$ can be written as a direct multiplicity-free integral of irreducible spherical (i.e., having a distinguished cyclic $K$-invariant vector) representations of $G$ indexed by the points of a set $\Omega$ defined in Sec. 1. Thus the equivalence class of $T_{zw}$ is uniquely determined by an equivalence class of measures on $\Omega$. The latter is called the spectral type of $T_{zw}$.

The spectral type of $T_{zw}$ substantially depends on whether or not $z$ and $w$ are integers. The case of noninteger $z$ and $w$ was studied in [1]. In the present paper, we consider the case where $z$ and $w$ are integers. The representation $T_{zw}$ is decomposed into a direct sum of subrepresentations $T_{pr}$, which we call blocks. We show that the spectral type of $T_{zw}$ is determined by a collection of measures on finite-dimensional faces $\Omega(p, q; r, s) \subset \Omega$ defined in Sec. 1. A similar result for the group $S(\infty)$ and the one-parameter family of representations $T_z$ for integer $z$ was obtained in [4].

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1. CHARACTERS AND THE GELFAND–TSETLIN GRAPH

In this section, we formulate several results concerning characters of arbitrary topological groups and, in particular, the infinite-dimensional unitary group. For the proofs of the results cited in Secs. 1 and 2, see [5] and references therein.

Definition 1. Let $K$ be a topological group. A character of $K$ is a continuous complex-valued function $\chi$ on $K$ satisfying the following three conditions:

(1) $\chi$ is central, i.e., constant on the conjugacy classes of $K$;
(2) $\chi$ is positive definite, i.e., for any finite number of elements $g_1, \ldots, g_n$ of $K$, the matrix $[\chi(g_j^{-1}g_i)]_{1 \leq i, j \leq n}$ is Hermitian and nonnegative definite;
(3) $\chi(e) = 1$, where $e$ is the identity of $K$.

The set of all characters of $K$ will be denoted by $\mathcal{X}(K)$. Clearly, $\mathcal{X}(K)$ is a convex set. Extreme points of this set are called extremal characters.

In this paper, $K$ is the infinite-dimensional unitary group $U(\infty) = \cup_{N \geq 1} U(N)$, which is the inductive limit of the unitary groups $U(N)$ consisting of unitary $N \times N$ matrices. The embedding of $U(N)$ into $U(N+1)$ is defined as follows: we identify $U(N)$ with the subgroup in $U(N+1)$ that consists of matrices leaving the $(N+1)$th basis vector fixed. Thus $U(\infty)$ is the group of infinite unitary matrices $U = [U_{ij}], i, j = 1, 2, \ldots$, with only finitely

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many entries $U_{ij}$ different from $\delta_{ij}$. We endow $U(\infty)$ with the inductive limit topology. In this topology, $U(\infty)$ is not a locally compact group.

The conjugacy classes of the group $U(N)$ are indexed by the spectra of unitary matrices, i.e., by the unordered collections of complex numbers $u_1, \ldots, u_N$ of absolute value 1. Then it is clear that the conjugacy classes of $U(\infty)$ are indexed by countable collections of complex numbers $(u_1, \ldots, u_n, \ldots)$ of absolute value 1 with only finitely many elements different from 1; the ordering of $u_i$ is irrelevant.

Denote by $\mathbb{R}^\infty$ the product of countably many copies of $\mathbb{R}$ and set

$$\mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R}.$$

Denote by $\Omega \subset \mathbb{R}^{4\infty+2}$ the subset of 6-tuples

$$\omega = (\alpha^+, \beta^+, \alpha^-, \beta^-, \delta^+, \delta^-)$$

such that

$$\alpha^+ = (\alpha_1^+ \geq \alpha_2^+ \ldots \geq 0) \in \mathbb{R}^\infty, \quad \beta^+ = (\beta_1^+ \geq \beta_2^+ \ldots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^\infty (\alpha_i^+ + \beta_i^+) < \delta^+, \quad \beta_1^+ + \beta_1^- \leq 1.$$

Let

$$\gamma^\pm = \delta^\pm - \sum_{i=1}^\infty (\alpha_i^+ + \beta_i^\pm) \geq 0.$$

For every $\omega \in \Omega$ we define a function $\chi^{(\omega)}$ on $U(\infty)$ as follows:

$$\chi^{(\omega)}(U) = \prod_{u \in \text{Spec } U} \left\{ e^{(\gamma^+(u-1) + \gamma^-(u-1))} \prod_{i=1}^\infty \frac{1 + \beta_i^+(u-1) + \beta_i^-(u-1) - 1 - \alpha_i^+(u-1) - \alpha_i^-(u-1)}{1 - \alpha_i^+(u-1) - 1 - \alpha_i^-(u-1)} \right\},$$

where $U$ is a matrix from $U(\infty)$ and the product ranges over all its eigenvalues. All but finitely many eigenvalues are equal to 1, so that the product over $u$ is in fact finite. The product over $i$ converges, since the sum of the parameters is finite. The numbers $\alpha_i^\pm, \beta_i^\pm,$ and $\gamma^\pm$ (or $\delta^\pm$) are called the Voiculescu parameters (see [6]).

**Theorem 1.1.** The functions $\chi^{(\omega)}$, where $\omega$ ranges over the set $\Omega$, are exactly the extremal characters of the group $U(\infty)$.

**Theorem 1.2.** For every character $\chi$ of the group $U(\infty)$ there exists a unique probability measure $P$ on the topological space $\Omega$ such that

$$\chi(U) = \int_\Omega \chi^{(\omega)}(U) P(d\omega), \quad U \in U(\infty).$$

The measure $P$ is called the spectral measure of $\chi$.

Let $p, q, r, s$ be integers. If all of them are nonnegative, then denote by $\Omega(p, q; r, s) \subset \Omega$ the subset of $\omega \in \Omega$ such that

$$\alpha_{p+1}^+ = \alpha_{p+2}^+ = \ldots = 0, \quad \beta_{q+1}^+ = \beta_{q+2}^+ = \ldots = 0,$$

$$\alpha_{r+1}^- = \alpha_{r+2}^- = \ldots = 0, \quad \beta_{s+1}^- = \beta_{s+2}^- = \ldots = 0,$$

$$\delta^+ = \sum (\alpha_i^+ + \beta_i^+), \quad \delta^- = \sum (\alpha_i^- + \beta_i^-).$$

If $p, r, s \geq 0, q < 0$, and $s \geq -q$, then by $\Omega(p, q; r, s) \subset \Omega$ we denote the subset of $\omega \in \Omega$ such that

$$\alpha_{p+1}^+ = \alpha_{p+2}^+ = \ldots = 0, \quad \beta_1^+ = \beta_2^+ = \ldots = 0,$$

$$\alpha_{r+1}^- = \alpha_{r+2}^- = \ldots = 0, \quad \beta_1^- = \ldots = \beta_q^- = 1, \quad \beta_{s+1}^- = \beta_{s+2}^- = \ldots = 0,$$

$$\delta^+ = \sum (\alpha_i^+ + \beta_i^+), \quad \delta^- = \sum (\alpha_i^- + \beta_i^-).$$

In the case where $p, r, q, s \geq 0, s < 0$, and $q \geq -s$, the sets $\Omega(p, q; r, s)$ are defined in a similar way. In all cases, the dimension of $\Omega(p, q; r, s)$ is equal to $p + q + r + s$.

The irreducible representations of the group $U(N)$ are indexed by ordered collections of $N$ integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$, called signatures. The number $N$ is called the length of the signature. The set of all