A GENERALIZATION OF A HARDY–LITTLEWOOD THEOREM

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We prove the following extension of a theorem by Hardy and Littlewood. Let $f$ be a holomorphic function in the unit disk and let

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^\frac{1}{p} = O(\varphi(r)), \quad r \to 1 - 0,$$

where $\varphi$ is a monotone increasing function on $(0, 1)$, and

$$\alpha_\varphi = \lim_{r \to 1 - 0} \frac{\varphi'(r)(1 - r)}{\varphi(r)}.$$

1. If $0 \leq \alpha_\varphi < +\infty$, then $M_p(r, f') = O(\frac{\varphi'(r)}{r})$, $r \to 1 - 0$;
2. if $\alpha_\varphi = +\infty$, then $M_p(r, f') = O(\varphi'(r))$, $r \to 1 - 0$. Bibliography: 4 titles.

Let $D = \{z \in C : |z| < 1\}$ be the unit disk in the complex plane $C$, let $H(D)$ be the set of holomorphic functions on $D$, and let $H^p$, $0 < p < +\infty$, be the Hardy class on $D$. The following statement had been proved in the paper [1].

**Theorem A.** Let $f \in H(D)$. If

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^\frac{1}{p} = O\left( \frac{1}{(1 - r)^\beta} \right),$$

then

$$M_p(r, f') = O\left( \frac{1}{(1 - r)^{\beta + 1}} \right).$$

The converse statement (estimate (2) implies estimate (1)) is valid as well.

It is natural to ask what form take estimates (1) and (2) if the power weight $\varphi(r) = \frac{1}{(1 - r)^\beta}$ is replaced by weights of a more general form, for example, by weights of the form

$$\varphi(r) = \frac{1}{(1 - r)^\beta} \left( \log \ldots \log \frac{A}{1 - r} \right)^j,$$

where $0 < \beta < +\infty$, $A > 0$, $0 < r < 1$, and $-\infty < j < +\infty$, or weights of the form

$$\varphi(r) = \exp \frac{1}{(1 - r)^\alpha} \quad \text{or} \quad \varphi(r) = \exp \ldots \exp \frac{1}{(1 - r)^\alpha},$$

where $\alpha > 0$ and $0 < r < 1$.

In this paper, we prove an analog of the Hardy–Littlewood theorem that covers all the above-mentioned cases.

**Theorem 1.** Let $f \in H(D)$. Assume that $M_p(r, f) \leq \varphi(r)$, where $0 < p < +\infty$ and $0 < r < 1$, and there exists the limit

$$\alpha_\varphi = \lim_{r \to 1 - 0} \frac{\varphi'(r)(1 - r)}{\varphi(r)}.$$

Then the following statements hold.

1. If $0 \leq \alpha_\varphi < +\infty$, then $M_p(r, f') = O\left( \frac{\varphi'(r)}{r} \right)$ as $r \to 1 - 0$.
2. If $\alpha_\varphi = +\infty$ and the function $\psi(x) = \log \varphi(1 - e^{-x})$ is convex and satisfies the condition

$$\frac{\psi''(x)}{\psi'(x)^2} = O(1)$$

as $x \to +\infty$, then the relation

$$M_p(r, f') = O(\varphi'(r))$$


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as \( r \to 1 - 0 \) holds for \( 1 \leq p < +\infty \). If \( 0 < p < 1 \), then
\[
M_p(r^2, f') = O(\varphi'(r))
\]
as \( r \to 1 - 0 \).

We start with two auxiliary statements.

**Theorem B** (see [2, p. 79]). If \( f \in H^p, \, 0 < p < +\infty \), then we can represent
\[
f(z) = f_1(z) + f_2(z),
\]
where \( f_j(z) \neq 0 \) for \( z \in D, \, j = 1, 2 \), \( f_1, f_2 \in H^p \), and the following estimate holds:
\[
||f_j||_{H^p} \leq 2||f||_{H^p}, \quad j = 1, 2.
\]

**Lemma.** Let \( \varphi \) be a monotonously increasing, positive function on \([0, 1]\) such that \( \varphi \in C^{(1)}[0, 1] \), and let \( 0 < \alpha_\varphi < +\infty \). Then the following estimates hold for any \( r \in [0, 1) \):
\[
\varphi \left( \frac{1 + r}{2} \right) \leq C \varphi(r) \quad \text{and} \quad \varphi \left( \frac{1 + \sqrt{r}}{2} \right) \leq C \varphi(r);
\]
here and in what follows, \( C \) is a constant whose value is not important for us.

**Proof.** The conditions of our lemma imply that, for an arbitrary \( \varepsilon > 0 \), there exists a value \( \rho_0 = \rho_0(\varepsilon) > 0 \) such that
\[
\frac{\varphi'(\rho)}{\varphi(\rho)}(1 - \rho) < (\alpha_\varphi + \varepsilon), \quad \rho_0 < \rho < 1.
\]
Integrating this inequality over the segment \([r, \frac{r + 1}{2}]\), \( \rho_0 < r < 1 \), we see that
\[
\log \frac{\varphi(\frac{1 + r}{2})}{\varphi(r)} < (\alpha_\varphi + \varepsilon) \log 2,
\]
which implies that
\[
\varphi \left( \frac{1 + r}{2} \right) < 2^{\alpha_\varphi + \varepsilon} \varphi(r).
\]
This proves the first estimate in (3). A similar reasoning establishes the second estimate. This completes the proof of our lemma. \( \square \)

Let us pass to the proof of Theorem 1.

First we assume that \( 1 \leq p \leq +\infty \).

Let \( 0 < r < \rho < 1 \). By the Cauchy formula,
\[
f'(z) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta, \quad z = re^{i\varphi}.
\]

Hence,
\[
|f'(re^{i\varphi})| \leq \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \frac{|f(\rho e^{i\varphi})|}{|\rho e^{i\varphi} - re^{i\varphi}|^2} d\theta = \frac{\rho}{2\pi} \int_{-\pi}^{\pi} \frac{|f(\rho e^{i(\varphi + \varphi')})|}{\rho^2 - 2r\rho \cos \theta + r^2} d\theta.
\]

We apply the Minkowski inequality to show that
\[
M_p(r, f') \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(\rho e^{i(\varphi + \varphi')})|^p}{\rho^2 - 2r\rho \cos \theta + r^2} d\theta \right)^{\frac{1}{p}} d\theta.
\]

This gives us the estimate
\[
M_p(r, f') \leq \frac{M_p(\rho, f)}{\rho^2 - r^2},
\]
for arbitrary \( 0 \leq r < \rho < 1 \).

By the condition of our theorem,
\[
M_p(\rho, f) \leq \varphi(\rho), \quad 0 < \rho < 1.
\]