PRODUCTS OF ORTHOPROJECTORS AND HERMITIAN MATRICES

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A proof of the following result is presented: A matrix $A \in M_n(\mathbb{C})$ can be represented as a product $A = PH$, where $P$ is an orthoprojector and $H$ is a Hermitian matrix, if and only if $A$ satisfies the equation $A^2 A = A^* A^2$ (the Radjavi-Williams theorem). Unlike the original proof, the new one makes no use of the Crimmins theorem. Bibliography: 2 titles.

1. Let $M_n(\mathbb{C})$ be the set of complex $n \times n$ matrices. The following proposition is Theorem 9 in [1]:

**Theorem 1.** For a matrix $A \in M_n(\mathbb{C})$ to be representable as a product

$$A = PH,$$ (1)

where $P$ is an orthoprojector and $H$ is a Hermitian matrix, it is necessary and sufficient that $A$ satisfy the equation

$$A^2 A = A^* A^2.$$ (2)

The proof of this theorem given in [1] relates it to the following result, due to T. Crimmins:

**Theorem 2.** For a matrix $A \in M_n(\mathbb{C})$ to be representable as a product

$$A = PQ,$$ (3)

where $P$ and $Q$ are orthoprojectors, it is necessary and sufficient that $A$ satisfy the equation

$$A^2 = AA^* A.$$ (4)

The aim of this short note is to present an alternative proof of Theorem 1, which does not use Crimmins’s result.

2. The necessity of the condition in Theorem 1 is trivially verified by substituting representation (1) into Eq. (2); then both sides turn out to be equal to the matrix $H^{\dagger} P^\dagger PH$.

In order to prove sufficiency, observe that the assertion of Theorem 1 is invariant with respect to unitary similarity transformations of $A$. Indeed, if

$$B = Q^* A Q, \quad Q Q^* = I_n,$$ (5)

then

$$B = (Q^* P Q)(Q^* H Q) = P_B H_B,$$ (6)

where, as above, $P_B$ is an orthoprojector, whereas $H_B$ is a Hermitian matrix. By substituting (3) into Eq. (2), we obtain an equation of the same type for $B$:

$$B^2 B = B^* B^2.$$ (7)

If we set

$$S_A = A^* A,$$ (8)

then (2) can be written as

$$A^* S_A = S_A A.$$ (9)

The above observation allows us to pass to a basis $e_1, \ldots, e_n$ with respect to which the Hermitian matrix (4) is diagonal:

$$S_A = \Lambda \oplus 0_s.$$ (10)

Here, $\Lambda$ is a diagonal $r \times r$ matrix with positive diagonal entries; $r = \text{rank}_A$, and $s = n - r$.

From representation (6) it is obvious that the vectors $e_{r+1}, \ldots, e_n$ form a basis in the subspace $\ker S_A$. Since $\ker A^* A = \ker A$, the last $s$ columns of $A$ also must be zero. On partitioning $A$ in conformity with (6), we obtain

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix}. $$ (11)
From (4) it follows that
\[ \Lambda = A_{11}^*A_{11} + A_{21}^*A_{21}. \] (8)

Relation (5) reduces to the equality
\[ A_{11}^* \Lambda = \Lambda A_{11}, \] (9)

which implies that \( \Lambda A_{11} \) is a Hermitian matrix. Then
\[ A_{11} \Lambda^{-1} = \Lambda^{-1}(\Lambda A_{11}) \Lambda^{-1} \] (10)

also is a Hermitian matrix.

3. Let \( A \) be an arbitrary matrix from \( M_n(\mathbb{C}) \) and let \( P \) be the orthoprojector onto the range \( \mathcal{L} \) of this matrix. Denote the orthoprojector onto the subspace \( \mathcal{L}^\perp \) by \( Q \), where \( Q = I_n - P \). Following [1], define the matrix
\[ H = PAP + AQ + QA^*. \] (11)

Then
\[ PH = PA = A. \] (12)

Indeed, we have
\[ PH = P^2AP + PAQ + (PQ)A^* = PAP + PAQ = PA(P + Q) = PA = A. \]

We apply relations (11) and (12) to the situation described in Theorem 1. The product representation of \( A \) (12) is the desired decomposition (1) if \( H \) is a Hermitian matrix. In other words, \( PAP \) must be a Hermitian matrix.

In the case of matrix (7), the orthoprojector onto its range is given by
\[ P = \begin{pmatrix} A_{11} & A_{21} \\ \Lambda^{-1} \end{pmatrix} \Lambda^{-1} \begin{pmatrix} A_{11}^* & A_{21}^* \\ \Lambda^{-1} \end{pmatrix} = \begin{pmatrix} A_{11} \Lambda^{-1}A_{11}^* & A_{11} \Lambda^{-1}A_{21}^* \\ A_{21} \Lambda^{-1}A_{11}^* & A_{21} \Lambda^{-1}A_{21}^* \end{pmatrix}. \]

It follows that
\[ PAP = AP = \begin{pmatrix} A_{11}^2 \Lambda^{-1}A_{11}^* & A_{11}^2 \Lambda^{-1}A_{21}^* \\ A_{21}A_{11} \Lambda^{-1}A_{11}^* & A_{21}A_{11} \Lambda^{-1}A_{21}^* \end{pmatrix}. \]

The blocks \( A_{11}^2 \Lambda^{-1}A_{11}^* = A_{11}(A_{11} \Lambda^{-1})A_{11}^* \) and \( A_{21}(A_{11} \Lambda^{-1})A_{21}^* \) are Hermitian because (10) is a Hermitian matrix. The blocks \( A_{11}^2 \Lambda^{-1}A_{21}^* \) and \( A_{21}A_{11} \Lambda^{-1}A_{11}^* \) are the Hermitian adjoints of each other because
\[ A_{11}^2 \Lambda^{-1}A_{21}^* = A_{11}(A_{11} \Lambda^{-1})A_{21}^* = A_{11}(\Lambda^{-1}A_{11}^*)A_{21}^* = (A_{21}A_{11} \Lambda^{-1}A_{11}^*)^*. \]

Thus, \( PAP \) is a Hermitian matrix, which implies that \( H \) is a Hermitian matrix as well. Theorem 1 is proved.

Remark. Let \( H_1 \) and \( H_2 \) be Hermitian matrices and let one of them be positive semidefinite. Then the eigenvalues of the product
\[ A = H_1 H_2 \]
are real. Moreover (see [2]), the nonzero eigenvalues of \( A \) are semisimple, and the Jordan blocks corresponding to the zero eigenvalue can only be of order one or two.

Under the assumptions of Theorem 1, the zero eigenvalue of \( A \) may be not semisimple already for \( n = 2 \). As an example, consider the Jordan block
\[ J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

It satisfies Eq. (2), both sides of which vanish on substituting \( A = J \). In the representation
\[ J = PH, \]
the orthoprojector \( P \) is of the form
\[ P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]
whereas as \( H \) one can choose an arbitrary matrix of the form
\[ \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \]
with a real entry \( x \).

Translated by Kh. D. Ikramov.