MATHEMATICAL MODELING OF NONLINEAR DYNAMICS IN
BISTABLE REACTION-DIFFUSION SYSTEMS WITH FRACTIONAL DERIVATIVES

B. Y. Datsko

We investigate the dynamics and conditions of emergence of complex autowave solutions in bistable re-
action-diffusion systems with time fractional derivatives. It is shown that fractional reaction-diffusion
systems have new properties in comparison with standard systems with derivatives of integer order. In
particular, in bistable systems with fractional derivatives, we have found new types of autowave solu-
tions, which cannot exist in standard reaction-diffusion systems. Results of the linear theory are sub-
stantiated by using computer simulation of a system with cubic nonlinearity, which enables us to simu-
late characteristic feedbacks and main types of autowave solutions. On the basis of the computational
experiment, we show that the order and relation between time fractional derivatives change qualitatively
the conditions of instability and nonlinear dynamics of bistable systems.

Introduction

In the last several decades, the investigation of reaction-diffusion (RD) equations extended substantially the
understanding of complex nonequilibrium phenomena in many systems of nature. On the basis of the mathe-
matical modeling of classical reaction-diffusion systems, many striking nonlinear phenomena in physical, bio-
logical, and chemical media were described and explained (see [1, 3, 13, 17]). Examples of these phenomena
are the formation of complex, dynamic, spatially inhomogeneous distributions of concentrations of reagents in
chemical reactions, ordered structures in colonies of microorganisms, propagation of pulses in nerve fiber and
cardiac muscle, complex immune biochemical reactions, and formation of domains of current density and elec-
tric fields in semiconductors and gases. In fractal, porous, viscous media, diffusion, as a rule, has an anomalous
nature [2, 16, 18, 19] and requires using the apparatus of differential equations with fractional derivatives to
model processes in them. Autowave processes were detected experimentally even in complex media such as
living cells and the intercellular medium [14], where diffusion demonstrates a substantially anomalous nature.
As a result, the investigation of RD systems with fractional derivatives [2, 7, 8, 11, 16, 21] has great prospects
from the standpoint of both discovery of new nonlinear effects and their practical application.

A derivative of integer order depends only on the local behavior of the function, whereas fractional deriva-
tives have a nonlocal nature and depend also on the preliminary behavior of the function. For this reason, dy-
namic systems with fractional derivatives are used first of all for modeling processes with memory. From this
standpoint, the investigation of autowave systems with fractional derivatives seems to be a particularly attractive
object in the modeling of self-organization phenomena because standard systems of this type, in turn, also have
their own certain type of memory in the form of space-time limit cycles. In combination with effects of memory
of nonlocal operators of fractional differentiation, this can lead both to a qualitative change in the dynamics of
the system and to new classes of nonlinear solutions (see [7–9]). In papers published in recent years [4–6], fea-
tures of nonlinear dynamics in basic monostable RD systems with fractional derivatives were analyzed. In the
present work, we investigate the nonlinear dynamics and conditions of emergence of complex autowave solu-
tions in bistable systems.
Statement of the Problem

In the general case, the mathematical reaction-diffusion model can be described by a system of \( m \) nonlinear partial differential equations of parabolic type

\[
\tau \cdot u_t = D \Delta u + f(u, A),
\]

in a space-time domain \( \Omega \times \Psi \) at given boundary conditions on \( \partial \Omega \times \Psi \) and initial conditions for \( r \in \Omega \), where \( \Omega \) determines the bounded space domain in \( \mathbb{R}^p \), \( p \in \{1, 2, 3\} \), with a smooth boundary \( \partial \Omega \), and \( \Psi = (0, T) \) is the time interval \( 0 < T < \infty \). In Eq. (1), the vector \( \tau \cdot 1 \) and matrix \( D_{m \times m} \) determine the time and space scales of the system, \( A \in \mathbb{R} \) is an external bifurcation parameter, \( f = (f_1, f_2, \ldots, f_m)^\top \) is the vector of given smooth functions of the kinetics of the reaction, and \( f_i(u_1(r, t), u_2(r, t), \ldots, u_m(r, t)) : \mathbb{R}^m \to \mathbb{R}^m \).

To investigate the qualitative features of self-organization phenomena, we can reduce the general system (1) to the basic system of two nonlinear equations with a positive feedback and a negative feedback, respectively \([1, 3, 13, 17]\)

\[
\tau_1 \frac{\partial u_1(x, t)}{\partial t} = \ell_1^2 \frac{\partial^2 u_1(x, t)}{\partial x^2} + W(u_1, u_2, A),
\]

\[
\tau_2 \frac{\partial u_2(x, t)}{\partial t} = \ell_2^2 \frac{\partial^2 u_2(x, t)}{\partial x^2} + Q(u_1, u_2, A),
\]

where \( \tau_1, \tau_2, \ell_1, \) and \( \ell_2 \) are characteristic times and lengths of the system, and \( W(u_1, u_2, A) \) and \( Q(u_1, u_2, A) \) are nonlinear functions of the variables \( u_1 \) and \( u_2 \) (activator and inhibitor, respectively) and a certain external parameter \( A \).

In system (2), (3), for periodic

\[
\left. u_i \right|_{x=0} = u_i \left|_{x=L}, \quad \left. \frac{\partial u_i}{\partial x} \right|_{x=0} = \left. \frac{\partial u_i}{\partial x} \right|_{x=L}, \quad i = 1, 2,
\]

or neutral boundary conditions

\[
\left. \frac{\partial u_i}{\partial x} \right|_{x=0} = \left. \frac{\partial u_i}{\partial x} \right|_{x=L} = 0, \quad i = 1, 2,
\]

there can exist spatially homogeneous stationary solutions that satisfy the system of algebraic equations

\[
W(u_1, u_2, A) = 0, \quad Q(u_1, u_2, A) = 0,
\]

the number of real solutions of which classifies system (2), (3) as monostable or multistable. The existence of the positive feedback in the first variable \( u_1 \)