REACHABLE SETS OF A LAMÉ TYPE DYNAMICAL SYSTEM

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By a version of the boundary control method (Belishev, 1986), a Riemannian manifold is recovered via its dynamical boundary inverse data, which correspond to the scalar wave equation, with the help of the virtual sources. We extend this version to the dynamical vector Lamé-type system. Such an extension is based on studying the structure of the reachable sets. The prospective goal of our study is to solve the inverse problem of recovering parameters of the Lamé system from the dynamical boundary data. Bibliography: 13 titles. Illustrations: 3 figures.

1 Introduction

In the case of the scalar wave equation, the problem of reconstructing a metric of a manifold from the boundary data was treated by using the version [1] of the boundary control method [2] with the help of the so-called virtual sources. From the physical point of view, the reconstruction procedure is reduced to localization of waves generated by the boundary sources (controls). The localization takes place in a small domain of a special shape, namely, a “cap” located near the end of the ray emanating at the boundary along the normal. Such a localization is possible due to an adequate choice of boundary controls [1, 3, 4].

The perspective goal of this paper is to extend the version [1] of the boundary control method to multi-velocity dynamical systems described by the vector wave equation. In such systems there are wave modes propagating with different velocities and interacting with each other, which makes the structure of reachable sets more complicated than that in the scalar case.

Let us describe the main result. We consider a Lamé type dynamical system with modes of

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two types ($p$-waves and $s$-waves) and restrict ourselves to the case of constant density ($\rho = 1$). The mode velocities $c_p$, $c_s$ depend on the points and $c_p > c_s$ everywhere.

The main result of this paper concerns the following question. For a Lamé type system we reproduce all the steps of the “scalar” procedure [1] providing the localization of waves in the cap. What does such a procedure give us in our case?

The answer is as follows. We show that, in the vector case (a Lamé type system), there are two caps: one each at the ends of the $p$-ray and $s$-ray. In each cap, the corresponding mode is localized: potential fields in the $p$-cap and solenoidal fields in the $s$-cap. Such a separation of caps is an encouraging fact for the inverse problem of reconstructing the Lamé coefficients from the dynamical boundary data. Hopefully, this fact is also true for the full Lamé system. If this is the case, one can reconstruct coefficients by using an analog of the tools developed in [1, 3, 4] in a time-optimal way.

Let us emphasize the new points of the obtained result. The above-mentioned inverse problem for Lamé type systems was solved in [5]. In such systems, the wave field splits into potential and solenoidal components. This fact is essential for the approach of [5], where boundary controls are “sorted” into two classes so that controls of each class generate only $p$-waves or only $s$-waves respectively. In this paper, we avoid such a sorting, which can be regarded as a certain progress. The splitting of waves is still essential, but is used only for analyzing the structure of reachable sets and the action of the corresponding projections.

2 Geometry

2.1. Metric. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth\(^1\) boundary $\Gamma$. In $\overline{\Omega}$, a smooth function (velocity) $c = c(x) > 0$ is given. It determines in $\overline{\Omega}$ the conformal Euclidean metric

$$d\tau^2 := \frac{|dx|^2}{c^2}, \tag{2.1}$$

where $|dx|$ is the Euclidean element of length in $\mathbb{R}^3$. We denote by $\tau(x, y)$ the distance relative to this metric. We call $T^* := \max_\Omega \tau(\cdot, \Gamma)$ the filling time.

For a subset $A \subset \overline{\Omega}$ we introduce its metric neighborhoods

$$\Omega^r[A] := \{x \in \overline{\Omega} \mid \tau(x, A) < r\}, \quad r > 0,$$

and denote by $\Omega^r := \Omega^r[\Gamma]$ the neighborhood of the boundary (the boundary layer of width $r$). The term “filling time” is motivated by the equality $T^* = \inf \{r > 0 \mid \Omega^r = \Omega\}$.

For $A \subset \overline{\Omega}$ we introduce the equidistant surface

$$\Gamma^r[A] := \{x \in \overline{\Omega} \mid \tau(x, A) = r\}, \quad r > 0$$

and denote by $\Gamma^r := \Gamma^r[\Gamma]$ the equidistance of the boundary.

With a point $x \in \overline{\Omega}$ we associate the set $\gamma(x) := \{\gamma \in \Gamma \mid \tau(x, \gamma) = \tau(x, \Gamma)\}$ of the nearest boundary points. It is known that for sufficiently small $r > 0$ and any $x \in \Omega^r$ the set $\gamma(x)$ consists of a single point and the system of semigeodesic (radial) coordinates with base $\Gamma$ is

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\(^1\) Throughout the paper, smooth surfaces, functions, fields etc. mean the smoothness of class $C^\infty$. 

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