FAN, SPLINT, AND BRANCHED RULES

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A splint of a root system for a simple Lie algebra appears naturally in studies of (regular) embeddings of reductive subalgebras. A splint can be used to construct branching rules. We demonstrate that implementation of splint properties drastically simplifies calculations of branching coefficients. Bibliography: 5 titles.

1. Introduction

An embedding $\phi$ of a root system $\Delta_1$ into a root system $\Delta$ is a bijective map of roots of $\Delta_1$ to a (proper) subset of $\Delta$ that commutes with vector composition law in $\Delta_1$ and $\Delta$,

$\phi : \Delta_1 \rightarrow \Delta,$

$\phi \circ (\alpha + \beta) = \phi \circ \alpha + \phi \circ \beta, \quad \alpha, \beta \in \Delta_1.$

Note that the image $\text{Im}(\phi)$ must not inherit the root system properties with the exception of addition rules equivalent to the addition rules in $\Delta_1$ (for pre-images). Two embeddings $\phi_1$ and $\phi_2$ can splinter $\Delta$ when the latter can be represented as a disjoint union of images $\text{Im}(\phi_1)$ and $\text{Im}(\phi_2)$. The term splint was introduced by D. Richter in [1], where a classification of splints for simple Lie algebras was obtained. It was also mentioned there that a splint must have tight connections with the injection fan construction. A fan $\Gamma \subset \Delta$ was introduced in [2] as a subset of a root system describing recurrent properties of branching coefficients for maximal embeddings. Injection fan is an efficient tool to study branching rules. Later this construction was generalized to nonmaximal embeddings and infinite-dimensional Lie algebras in [3, 4].

In the present paper, we study connections between splint and injection fan for a regular embedding of reductive subalgebras $a$ in a simple Lie algebra $\mathfrak{g}$. We show that (under certain conditions described in Sec. 3), splint is a natural tool to study reduction properties of $\mathfrak{g}$-modules with respect to a subalgebra $a \hookrightarrow \mathfrak{g}$. Using this tool, we obtain the main result, a one-to-one correspondence between weight multiplicities in irreducible modules of splint and branching coefficients for a reduced module $L^\mu_{\mathfrak{g} \setminus \mathfrak{a}}$.

2. Injections and Splints

Consider a simple Lie algebra $\mathfrak{g}$ and its regular subalgebra $a \hookrightarrow \mathfrak{g}$ such that $a$ is a reductive subalgebra $a \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^+_a \subset \mathfrak{h}^+_\mathfrak{g}$. Let $a^\circ$ be a semisimple summand of $a$; this means that $a = a^\circ \oplus (1 + a^\circ) \oplus \ldots$. We consider $a^\circ$ that is a proper regular subalgebra and $a$ that is the maximal subalgebra with $a^\circ$ fixed, i.e., the rank $r$ of $a$ is equal to that of $\mathfrak{g}$.

The following notation is used:

$r, (r_a^\circ)$ is the rank of $\mathfrak{g}$ (of $a^\circ$, respectively);

$\Delta, (\Delta_a^\circ)$ is the root system;

$\Delta^+ (\Delta_a^+, \text{ respectively})$ is the positive root system (of $\mathfrak{g}$ and $a$, respectively);

$S_a, (S_a)$ is the system of simple roots (of $\mathfrak{g}$ and $a$, respectively);

$\alpha_i, (\alpha_{a(i)}^j)$ is the $i$th ($j$th, respectively) simple root for $\mathfrak{g}$ ($a$, respectively);

$i = 0, \ldots, r, j = 0, \ldots, r_a^\circ$;

$\omega_i, (\omega_{a(i)}^j)$ is the $i$th ($j$th, respectively) fundamental weight for $\mathfrak{g}$ ($a$, respectively);

$i = 0, \ldots, r, j = 0, \ldots, r_a^\circ$;

$W (W_a)$ is the corresponding Weyl group;

$C (C_a)$ is the fundamental Weyl chamber;

$C (C_a)$ is the closure of the fundamental Weyl chamber;

$\epsilon (\omega) := (-1)^{\text{length}(\omega)}$;

$\rho (\rho_a)$ is the Weyl vector;

$L^\mu (L^\mu_a)$ is the integrable module of $\mathfrak{g}$ with the highest weight $\mu$.

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(integrable $a$-module with the highest weight $\nu$, respectively); $\mathcal{N}^\mu (\mathcal{N}^\nu_a)$ is the weight diagram of $L^\mu (L^\nu_a$, respectively); $P$ ($P_a$, respectively) is the weight lattice; $P^+$ ($P^+_a$, respectively) is the dominant weight lattice; $\mathcal{E}$ ($\mathcal{E}_a$, respectively) is the formal algebra; $m^\mu_\xi (m^\nu_\xi)$ is a multiplicity of the weight $\xi \in P$ ($\xi \in P_a$, respectively) in the module $L^\mu$ ($L^\nu_a$, respectively); $\mathrm{ch} (L^\mu)$ $\mathrm{ch} (L^\nu_a)$, respectively) is the formal character of $L^\mu (L^\nu_a$, respectively); $\mathrm{ch} (L^\mu) = \sum_{w \in W} c(w)e^{w(\mu + \rho) - \rho}$ is the Weyl formula; $R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) (R_a := \prod_{\alpha \in \Delta^+_a} (1 - e^{-\alpha})$, respectively) is the Weyl denominator.

Let $L^\mu$ be completely reducible with respect to $a$, $L^\mu_{\mathfrak{g}|a} = \bigoplus_{\nu \in P^+_a} b^\nu L^\nu_a$, where $\pi_a c h (L^\mu) = \sum_{\nu \in P^+_a} b^\nu \mathrm{ch} (L^\nu_a)$. 

(1)

For the modules we are interested in, the Weyl formula for $\mathrm{ch} (L^\mu)$ can be written in terms of singular elements [5]: $\Psi^{(\mu)} := \sum_{w \in W} \epsilon (w)e^{w(\mu + \rho) - \rho}$, namely, $\mathrm{ch} (L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(\nu)}} = \frac{\Psi^{(\mu)}}{R}$. 

(2)

The same is true for the submodules $\mathrm{ch} (L^\nu_a)$ in (1): $\mathrm{ch} (L^\nu_a) = \frac{\Psi^{(\nu)}_a}{\Psi^{(\nu)}_a} = \frac{\Psi^{(\nu)}_a}{R_a}$, where $\Psi^{(\nu)}_a := \sum_{w \in W_a} \epsilon (w)e^{w(\nu + \rho_a) - \rho_a}$.

(3)

Applying formula (2) to the branching rule (1), we get a relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi^{(\nu)}_a$:

$$\sum_{w \in W} \epsilon (w)e^{w(\mu + \rho) - \rho} = \frac{\sum_{\nu \in P^+_a} b^\nu \sum_{w \in W_a} \epsilon (w)e^{w(\nu + \rho_a) - \rho_a}}{\prod_{\beta \in \Delta^+_a} (1 - e^{-\beta})},$$

$$\frac{\Psi^{(\mu)}}{R} = \frac{\Psi^{(\nu)}_a}{R_a},$$

(3)

In [3], it was proven that singular branching coefficients $k^{(\mu)}_\xi$ corresponding to the injection $a \hookrightarrow \mathfrak{g}$ are subject to the following set of recurrent relations:

$$k^{(\mu)}_\xi = -\frac{1}{s (\gamma_0)} \sum_{u \in W/W} \epsilon (u) \dim (L^{\mu_{a \perp} (u)}) \delta_{\xi - \gamma_0, \pi_\xi (u(\mu + \rho) - \rho)} + \sum_{\gamma \in \Gamma_{\frac{\mathfrak{g}}{\mathfrak{a}}}^{\perp} \pi_{\xi + \gamma}} s (\gamma + \gamma_0) k^{(\mu)}_{\xi + \gamma},$$

(4)

where $a_{\perp}$ is the subalgebra determined by the roots of $\mathfrak{g}$ orthogonal to roots of $a$ and $W_{\perp}$ is a Weyl group of $a_{\perp}$,

$$\Delta_{a_{\perp}} := \left\{ \beta \in \Delta_+ \mid \text{for all } h \in \mathfrak{h}_a; \beta (h) = 0 \right\},$$

$$\tilde{a}_{\perp} := a_{\perp} \oplus \mathfrak{h}_{\perp}, \quad \tilde{a} := a \oplus \mathfrak{h}_{\perp},$$

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