RANDOM VARIABLES ASSOCIATED WITH THE FAREY TREE

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The paper investigates random variables associated with the limit distribution for the Farey tree rationals. Bibliography: 8 titles.

INTRODUCTION

The present paper is closely related to [1], where a smooth strictly convex curve containing a large number of rational points with a fixed denominator was constructed. All the main results below (Theorems 1–4) are devoted to studying the curvature of this curve.

In estimating the number of rational points with a fixed denominator on a fixed convex curve, the question on its smoothness is of considerable importance.

By $N(k)$ denote the number of rational points with denominator $k$ lying on such a curve. In [2], it was proved that $N(k) = o(K^{2/3})$. Thus, the famous Janik result cannot hold in this statement.

Swinnerton-Dyer [3] demonstrated that $N(k) = O_e(k^{0.61+\varepsilon})$, where $\varepsilon > 0$ is arbitrarily small, if the curve is in the class $C^3$ and the third derivative is separated from zero. In [4], Bombieri and Pila obtained a result close to the improvable one under the assumption that the curve belongs to the class $C^\infty$. (Obviously, the improvable result is $N(k) = O(k^{1/2})$.)

In the case of a curve constructed in [1], $N(k) \approx k^{\log 2/\log 3}$ for the sequence $k = 3^n$ (observe that $\log 2/\log 3 > 0.639$).

In [5], for a very sparse sequence of $k$, curves with greater $N(k)$ are constructed, but the latter curves are not smooth.

The curve from [1] is given by a pair of functions dependent on the parameter $t$ (for the definitions of the functions $P(t)$ and $Q(t)$, see below).

These functions were obtained with the use of weighted means of the numerators and denominators of rationals of a fixed level of the Farey tree (see [6]).

The functions satisfy the relation $dP/dQ = t$, implying that $d^2 P/dQ^2 = 1/dQ/dt$.

As has been mentioned above, the main results of the present paper are related to studying $dQ/dt$.

In particular, it is shown that $dQ/dt = 0$ almost everywhere with respect to the measure $dt$. In addition, specific points at which $dQ/dt = 0$ are indicated (they are all rationals and a sufficiently large number of quadratic irrationals).

Also an infinite set of points such that $dQ/dt = \infty$ is presented. The latter set contains, in particular, all points of the Markov spectrum.

1. Farey tree. Functions $P(t)$ and $Q(t)$

One method for constructing the Farey tree is as follows: Points of the zero level are the points 0/1 and 1/1. The points of level $n$ are obtained from the points of level $n-1$ with the use of Farey mediants, i.e., if $a/b$ and $c/d$ are two adjacent points of level $n-1$, then in between them the point $(a+c)/(b+d)$ of level $n$ is inserted.

The points of level $n$ can also be described in a different way by using continued fractions. New points of level $n$ (i.e., those which do not belong to lower levels) are exactly those rational numbers $r$ whose continued fraction expansions, $r = [0, k_1, \ldots, k_l]$, satisfy the condition $k_1 + \cdots + k_l = n+1$ (cf. [6]).

The set of all points of level $n$ is denoted by $W_n$.

Now we introduce the functions $P(t)$ and $Q(t)$ for $0 \leq t \leq 1$.

Let $P_n(t)$ be the step function defined as follows:

$$P_n(t) = \frac{1}{3^n} \sum_{a/b \in W_n, a/b \leq t} a.$$
In [1], it was demonstrated that \( \lim_{n \to \infty} P_n(t) \) exists for all \( t \).
Denote this limit by \( P(t) \). One can prove that \( P(t) \) is a continuous and strictly increasing function. In addition, \( P(0) = 0 \) and \( P(1) = 1/2 \).

The function \( Q(t) \) is constructed similarly, with the difference that it takes into account the distribution of the denominators of the rationals in \( W_n \),

\[
Q_n(t) = \frac{1}{3^n} \sum_{\substack{a/b \in W_n \\ a/b \leq t}} b
\]

and

\[
Q(t) = \lim_{n \to \infty} Q_n(t).
\]

The function \( Q(t) \) also is continuous and strictly increasing, and, in addition, \( Q(0) = 0 \), \( Q(1) = 1 \).

It is clear that the parametric function \((P(t), Q(t))\) is continuous, and \( P \), as a function of \( Q \), is strictly increasing.

Moreover, it is continuously differentiable, and \( dP/dQ = t \). The latter relation implies that \( d^2 P/dQ^2 = 1/4 Q \).

The curve \((P(t), Q(t))\) is strictly convex for \( 0 \leq t \leq 1 \).

**Remark.** In [1], the latter assertion was proved by a lengthy computation, involving treatment of a large number of different cases. Actually, it immediately follows from the fact that \( dP/dQ \) is monotone increasing.

State Theorems 1–4.

**Theorem 1.** For almost all \( t \) (\( 0 < t < 1 \)) in the Lebesgue measure, \( Q'(t) = 0 \).

**Remark.** Thus, the function \( Q(t) \) provides an example of a strictly increasing function whose derivative vanishes almost everywhere.

**Theorem 2.** Let \( r \) be a rational. Then \( Q'(r) = 0 \).

**Remark.** Theorem 2 was established in [1]. For completeness, here we present a more detailed proof.

**Theorem 3.** Let \( d \) be a sufficiently large square-free number and let \( \alpha \in \mathbb{Q}(\sqrt{d}) \). Then there is a class \( C \) of irrationals such that \( Q'(\alpha) = 0 \) for all \( \alpha \in C \).

**Theorem 4.** Let \( t \) be an irrational number whose continued fraction expansion is of the form \( t = [0, k_1, \ldots, k_l, \ldots] \), where, for every \( l \), either \( k_l = 1 \) or \( k_l = 2 \). Then \( Q'(t) = \infty \).

2. PROOFS OF THEOREMS 1–4

First, we prove a few auxiliary results related to bounding \( Q(t_1) - Q(t_2) \) from above and below.

**Lemma 1.** Let \( r_1 = p_1/q_1 \) and \( r_2 = p_2/q_2 \) be adjacent points in \( W_N \). Then

\[
|Q(r_1) - Q(r_2)| = \frac{q_1 + q_2}{2 \cdot 3^N}.
\]

**Proof.** In [1], it was demonstrated that

\[
Q(r) = Q_n(r) - \frac{q + 1}{2 \cdot 3^n},
\]

where \( r = p/q \) is a rational, \( n \) is an arbitrary integer satisfying the condition \( n \geq n_0 \), and \( n_0 \) is the exact level of \( r \) in the Farey tree.

Let, for definiteness, \( r_1 < r_2 \). Then, by the definition of the function \( Q_N(t) \), we have

\[
Q_N(r_2) - Q_N(r_1) = \frac{q_2}{3^n},
\]

because \( r_1 \) and \( r_2 \) are adjacent points in \( W_N \).

Substituting this relation into (1), we arrive at the lemma assertion for \( n = N \). \( \square \)

**Lemma 2.** Let \( t_1 = [0, k_1, \ldots, k_s, m, \ldots] \) (\( t_1 \) is allowed to be rational) and \( t_2 = [0, k_1, \ldots, k_l, u, \ldots] \), where \( u < m \) (\( t_2 \) is also allowed to be rational). Let \( n = k_1 + \cdots + k_i \). Then

\[
|Q(t_2) - Q(t_1)| \leq C \frac{q u}{3^{n+u}},
\]

where \( C \) is an absolute constant, and \( q \) is determined from the relation \( r = p/q = [0, k, \ldots, k_i] \).