WAVE WALLS FOR WAVES ON THE SURFACE OF A HEAVY LIQUID

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An asymptotic analysis of surface water waves is made. The gradient of the ocean depth is assumed to be small. An ansatz for the asymptotic expansion in this small parameter is suggested. The goal is to construct the complete asymptotic expansion for a solution localized near a moving line on the liquid surface. The approach is based on the space-time ray method. The eiconal and amplitudes are found in the form of formal power series. Chains of recurrent equations for their terms are derived. Their solvability is proved. The first term is obtained in explicit form. Bibliography: 11 titles.

1. INTRODUCTION

Waves on the surface of an ocean with slowly varying depth are considered. Solutions of the wave wall type are built. A wave wall is a packet of short waves, which can be shown in the following way (see Fig. 1).

Fig. 1. A wave wall.

Asymptotic methods are used in solving the problem. It is assumed that the depth of an ocean varies slowly. The rate of change of the depth is characterized by a small parameter $\varepsilon$. The waves that form a wave packet correspond to the principal terms of the asymptotic expansion of the solution with respect to $\varepsilon$. The size of the domain where the wave packet is different from 0 significantly has order $O(\sqrt{\varepsilon})$. It is assumed that the size of this domain is much less than the curvature radius of the wave front (this condition ensures the absence of the overlapping of neighbor packets).

In space-time, a solution of the wave wall type is concentrated in a neighborhood of the surface “woven” of space-time rays.

Similar problems were previously considered in papers [1–6]. In particular, in [1] the method due to V. P. Maslov of complex Lagrange manifolds was used, and in [2] a solution of the wave film type was constructed. In the present paper, we apply the space-time ray method (see [3–6]).

2. INITIAL EQUATIONS

For completeness of presentation, we give some initial formulas. The derivation of these formulas is described in detail in [6].

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Let a heavy liquid fill an ocean
\[
\begin{align*}
-\infty & < x^1, \quad x^2 < \infty; \\
\end{align*}
\tag{2.1}
\]
here, \( z = -H \) is the surface of the ocean floor, \( H = H(x^1, x^2); \) \( \vec{v} = \vec{v}(t, x^1, x^2, z) \) is the velocity of the liquid. Let the liquid be incompressible, i.e., \( \text{div} \vec{v} = 0. \) Let the flow be irrotational, i.e., \( \vec{v} = -\nabla \Phi. \) Then the potential of the velocity field \( \Phi \) is a harmonic function:
\[
\Delta \Phi = 0. \tag{2.2}
\]
We want to clarify whether waves of some definite type may exist on the surface of the liquid. We assume that the floor \( z = -H(x^1, x^2) \) is impermeable:
\[
(\vec{v}, n) \bigg|_{z = -H} = 0, \tag{2.3}
\]
which is equivalent to the condition
\[
\left( \frac{\partial \Phi}{\partial z} + \frac{\partial H}{\partial x^1} \frac{\partial \Phi}{\partial x^1} + \frac{\partial H}{\partial x^2} \frac{\partial \Phi}{\partial x^2} \right) \bigg|_{z = -H} = 0. \tag{2.4}
\]
The second boundary condition has the form
\[
\left( \frac{\partial^2 \Phi}{\partial \tau^2} + g \frac{\partial \Phi}{\partial z} \right) \bigg|_{z = 0} = 0, \tag{2.5}
\]
where \( g \) is the acceleration of gravity. Thus, we shall work with Eq. (2.2) under boundary conditions (2.4) and (2.5).

3. The space-time ray method

We give the main statements of the space-time ray method. It is a known and frequently encountered method for solving problems of such a kind. One can find a detailed description of it in paper [3]. We shall need it in further constructions. So, we solve the problem of propagation of a surface wave over a weakly curved floor. Let the depth \( H \) be a slowly varying function of the coordinates \( x^1, x^2: \)
\[
H = H(\varepsilon x^1, \varepsilon x^2), \tag{3.1}
\]
where \( \varepsilon \ll 1 \) is a small parameter of the problem. Introduce slowly varying coordinates and time:
\[
\tau = \varepsilon t, \quad \xi^1 = \varepsilon x^1, \quad \xi^2 = \varepsilon x^2. \tag{3.2}
\]
The ansatz of the space-time ray (STR) method is an expansion of the form
\[
\Phi \sim e^{\frac{2\pi j_{\xi^1} \xi^2}{\varepsilon}}, \quad \sum_{j=0}^{\infty} \Phi_j(\tau, \xi^1, \xi^2, z)(\varepsilon \xi^2)^j, \tag{3.3}
\]