A “COMPLEX SOURCE” IN THE 2D REAL SPACE

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The paper concerns a complexified Green’s function $g_*$ for the 2D Helmholtz equation, which is studied as a nonparaxial model of a Gaussian beam. The function $g_*$ satisfies a certain nonhomogeneous Helmholtz equation in the real space with a source distribution dependent on a choice of branch of a certain complex square root. Various choices of branch cut are discussed, and the corresponding source distribution is calculated. Bibliography: 13 titles.

1. Introduction

Subsequent to asymptotic constructions of solutions of the Helmholtz equation localized for high frequencies, which were called Gaussian beams (for example, see [1]), interest aroused in exact solutions having a similar asymptotic behavior. It was motivated by the fact that asymptotic solutions are suitable only in a small neighborhood of the axis of a beam [2, 3]. Another motive (see [4, 5]) was the quest for statement of diffraction problems without appealing to approximate solutions. As opposite to the asymptotic theory, which easily allows a smooth inhomogeneity of a medium [1], the construction of exact solutions requires constant coefficients (see [6]).

In papers [2, 3], it was considered a complexified Green’s function of the three-dimensional Helmholtz operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2,$$

which has the form $G_* = \exp(ikR_*)/R_*$, where $R_* = \sqrt{x^2 + y^2 + (z - ia)^2}$ and $a > 0$ is a free parameter. The function $G_*$, which is called the field of a complex source (see [2]), for $ka \gg 1$ has a behavior typical of a Gaussian beam. It is important (this was first noted in [2]) that the function $R_*$ is many-valued in $\mathbb{R}^3$. Its uniquely determined branch (and, thus, a branch of $G_*$) has a jump on a surface $\bar{S}$ with boundary $\{x^2 + y^2 = a^2, \; z = 0\}$, which is determined by the choice of a cut. The function $G_*$ satisfies a nonhomogeneous Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)G_* = \bar{F}$$

with a source function $\bar{F}$ on the right-hand side, the support of which lies on $\bar{S}$. Both the form of the function $\bar{F}$ and the asymptotic behavior of $G_*$ depend on the choice of a cut. These questions for the 3D case were studied in detail in [7].

In the present note, we consider a similar question in the 2D case. Interest in the 2D “complex source” can be traced beginning with paper [5], where constructions based on a complexified Green’s function for the 2D Helmholtz operator were applied to solving the diffraction problem of Gaussian beams on a plane interface of media. The 2D case has specific features as compared with the 3D case (see [7]). Here, in distinction to the 3D case, the function $g_*$ has logarithmic branching, and in computing the source function, the regularization of an integral divergent in a power way is not of necessity.

2. A complexified Green’s function

The Green’s function for the two-dimensional Helmholtz equation, which corresponds to a divergent cylindrical wave (for the time dependence $e^{-i\omega t}$), has the form

$$g = -\frac{i}{4}H_0^{(1)}(kr), \quad r = \sqrt{x^2 + z^2},$$

where $H_0^{(1)}$ is the Hankel function. We complexify it, shifting by an imaginary constant in the variable $z$,

$$g_* = -\frac{i}{4}H_0^{(1)}(kr_*), \quad r_* = \sqrt{x^2 + (z - ia)^2},$$

where $a > 0$. The function $g_*$ is not single-valued in $\mathbb{R}^2$. It satisfies the nonhomogeneous equation

$$(\Delta + k^2)g_* = F(x, z), \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2, \quad k > 0$$

with a certain distribution $F$ on the right-hand side. The function of the source $F$ depends on a choice of branch of $g_*$. 

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As in [7], to determine a branch it is convenient to us to consider the plane of the complex variable

\[ w = u + iv = x^2 + z^2 - a^2 - 2iaz. \]  

(3)

The argument of \( H_0^{(1)}(\gamma) \) in (1) is \( k\sqrt{w} \). At the point \( w = 0 \), the function \( g_* \) has branching of the type \( \ln \sqrt{w} \).

For real \( x \) and \( z \), the variable \( w \) takes values in the domain

\[ \Pi = \left\{ \frac{u}{4\alpha^2v^2 - a^2} \right\} \]

(4)
depicted in Fig. 1. To each inner point \( w = u + iv \) of the domain \( \Pi \) there corresponds a one-to-one way a pair of points in \( \mathbb{R}^2 \) symmetric about the axis \( z \). From (3) one can easily obtain their coordinates:

\[ x = \pm \sqrt{u + a^2 - (v/2a)^2}, \quad z = -v/2a. \]  

(5)

To the boundary \( \partial \Pi \) of the domain there corresponds the axis \( z \). The points \( x = \pm a, z = 0 \) correspond to the branch point \( w = 0 \).

On the \( w \) plane, we draw a cut along a certain curve \( \gamma \). Denote by \( S \) the corresponding curve in \( \mathbb{R}^2 \) with endpoints \( \{x = \pm a, z = 0\} \). Obviously, \( S \) is symmetric about the axis \( z \) and is not necessarily connected. If \( \gamma \) intersects \( \partial \Pi \), then the curve \( S \) intersects the axis \( z \). The function \( g_* \) has a jump on \( S \) and \( \text{supp} F = S \).

In addition to the choice of a cut, we need to choose a branch of \( g_* \). We always choose it in such a way that, as \( z \to +\infty \), \( g_* \) corresponds to the outgoing wave. In [8], it was noted that there exist two essentially different situations of choices of cuts and branches. In one case, we have an outgoing wave that describes a Gaussian beam as \( z = +\infty \) and is damped as \( z = -\infty \). In the other case, we have a beam arriving from \( z = -\infty \) and outgoing to \( z = +\infty \). In [8], these two cases are called a source choice and a beam choice, respectively. We begin by studying the first case.

3. THE CASE OF A SOURCE CHOICE

First we take a cut along the negative real semiaxis. The segment \( S = \{ |x| \leq a, z = 0 \} \) corresponds to this cut. We take the branch of \( \ln \sqrt{w} \) real for \( \text{Re} w > 0, \text{Im} w = 0 \). For such a choice of branch of \( g_* \), we have the boundary values \( -\frac{1}{4} H_0^{(1)}(\pm \sqrt{x^2 - a^2}) \) for \( |x| \leq a, z = \pm 0 \).

3.1. Computation of the source function. We calculate the corresponding function \( F \), considering it in the usual way (see [9]) on the basic functions \( f = f(x, z) \),

\[ (F, f) = \iint_{\mathbb{R}^2} g_*(\Delta + k^2)f \, dx \, dz. \]  

(6)

Represent (6) in the form of the limit

\[ \lim_{\beta \to 0} \lim_{\Omega_{x,\beta} \to 0} \iint_{\Omega_{x,\beta}} g_*(\Delta + k^2) \, f \, dx \, dz, \]  

(7)

where the integral is taken over the domain \( \Omega_{x,\beta} \) shown in Fig. 2.