NONLINEAR DIFFERENCE EQUATIONS IN THE SPACES OF BOUNDED TWO-SIDED SEQUENCES

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We establish conditions for the existence of bounded solutions of nonlinear difference equations.

1. Main Functional Spaces and the Object of Investigation

Let $\mathbb{Z}$ be the set of all integers, let $\mathbb{R}$ be the set of all real numbers, let $\mathbb{C}$ be the set of all complex numbers, let $E_n$, $n \in \mathbb{Z}$, be arbitrary Banach spaces with norms $\| \cdot \|_{E_n}$, $n \in \mathbb{Z}$, and zero vectors $0_n$, $n \in \mathbb{Z}$, respectively, let $l_p$, $1 \leq p \leq \infty$, be Banach spaces of two-sided sequences $x = (x_n)_{n \in \mathbb{Z}}$, where $x_n \in E_n$, $n \in \mathbb{Z}$, with zero element $0 = (0_n)_{n \in \mathbb{Z}}$ for each of which

$$\sum_{n \in \mathbb{Z}} \| x_n \|_{E_n}^p < \infty \quad \text{for} \quad p \in [1, \infty)$$

and $\sup_{n \in \mathbb{Z}} \| x_n \|_{E_n} < \infty$ for $p = \infty$ with the norm

$$\| x \|_{l_p} = \begin{cases} \left( \sum_{n \in \mathbb{Z}} \| x_n \|_{E_n}^p \right)^{1/p} & \text{for} \quad p \in [1, \infty), \\ \sup_{n \in \mathbb{Z}} \| x_n \|_{E_n} & \text{for} \quad p = \infty, \end{cases}$$

respectively, let $C_0$ be the Banach space of two-sided sequences $x = (x_n)_{n \in \mathbb{Z}}$, $x_n \in E_n$, $n \in \mathbb{Z}$, with zero element $0 = (0_n)_{n \in \mathbb{Z}}$ for each of which $\lim_{n \to \infty} \| x_n \|_{E_n} = 0$ and the norm

$$\| x \|_{C_0} = \sup_{n \in \mathbb{Z}} \| x_n \|_{E_n}$$

(if all spaces $E_n$, $n \in \mathbb{Z}$, coincide with a Banach space $E$, then the spaces $l_p$, $1 \leq p \leq \infty$, and $C_0$ are denoted by $l_p(\mathbb{Z}, E)$, $1 \leq p \leq \infty$, and $c_0(\mathbb{Z}, E)$, respectively, and let $L(X, Y)$ be a Banach space of linear continuous operators $A$ acting from the Banach space $X$ into the Banach space $Y$ with the norm

$$\| A \|_{L(X, Y)} = \sup_{\| x \|_X = 1} \| Ax \|_Y.$$ 

Consider a difference equation

$$x_n = A_{n-1}x_{n-1} + F_n(x_{n-1}) + h_n, \quad n \in \mathbb{Z}, \quad (1)$$

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where \( h = (h_n)_{n \in \mathbb{Z}} \) is an element of one of the spaces \( l_p \), \( 1 \leq p \leq \infty \), or \( C_0 \) and the mappings \( A_n \in L(E_n, E_{n+1}) \) and \( F_n : E_{n-1} \to E_n \), \( n \in \mathbb{Z} \), are such that
\[
\sup_{n \in \mathbb{Z}} \| A_n \|_{L(E_n, E_{n+1})} < \infty,
\]
and, for any number \( r > 0 \),
\[
\sup_{n \in \mathbb{Z}, x \in E_{n-1}, \| x \|_{E_{n-1}} \leq r} \| F_n(x) \|_{E_n} < \infty.
\]

The aim of the present paper is to establish conditions for the existence of solutions of Eq. (1) in the spaces \( l_p \), \( 1 \leq p \leq \infty \), and \( C_0 \) (in the case \( h \in l_\infty \) and zero mappings \( F_n \), \( n \in \mathbb{Z} \). this equation is investigated in [1]).

In the investigation of Eq. (1), we use elements of the theory of \( c \)-continuous operators.

2. \( c \)-Continuous and \( c \)-Completely Continuous Operators

Consider operators
\[
P_m : l_\infty \to C_0, \quad m \in \mathbb{N}, \quad \text{and} \quad Q_n : l_\infty \to E_n, \quad n \in \mathbb{Z},
\]
given by the equalities
\[
(P_m x)_n = \begin{cases} 
   x_n & \text{for } |n| \leq m, \\
   0_n & \text{for } |n| > m,
\end{cases}
\]
and
\[
Q_n x = x_n.
\]

Let \( \mathcal{X} \) be either the space \( l_p \), \( 1 \leq p \leq \infty \), or \( C_0 \). A sequence of elements \( x_k \in \mathcal{X}, \ k \geq 1 \), is called \textit{locally convergent} to \( x \in \mathcal{X} \) as \( k \to \infty \) and denoted as follows:
\[
x_k \xrightarrow{\text{loc, } \mathcal{X}} x \quad \text{as} \quad k \to \infty
\]
if
\[
\sup_{k \geq 1} \| x_k - x \|_{\mathcal{X}} < \infty \quad \text{and} \quad \lim_{k \to \infty} \| P_m(x_k - x) \|_{\mathcal{X}} = 0
\]
for any number \( m \in \mathbb{N} \).

An operator \( \mathcal{H} : \mathcal{X} \to \mathcal{X} \) is called \textit{\( c \)-continuous} if, for any \( x \in \mathcal{X} \) and \( x_k \in \mathcal{X}, \ k \in \mathbb{N} \), such that \( x_k \xrightarrow{\text{loc, } \mathcal{X}} x \) as \( k \to \infty \), the relation
\[
\mathcal{H} x_k \xrightarrow{\text{loc, } \mathcal{X}} \mathcal{H} x \quad \text{as} \quad k \to \infty
\]
is true.