THE SPECTRUM OF A PERTURBED OPERATOR ASSOCIATED WITH A
HYPERBOLIC TORAL AUTOMORPHISM

A. M. Levin* UDC 517.984.5

In this paper, we consider a Markov operator (i.e., a contraction preserving the subspace of constants and the nonnegativity of functions) in the $L^2$ space on the $n$-dimensional torus that is a special perturbation of the unitary operator corresponding to a hyperbolic toral automorphism. We prove some properties of its spectrum and the spectra of some related operators. Bibliography: 4 titles.

1. MARKOV OPERATORS AND POLYMORPHISMS

Definition (see [2]). A Markov operator in the Hilbert space $L^2(X_0, \mu_0)$ of complex-valued square integrable functions on a Lebesgue–Rokhlin space $(X_0, \mu_0)$ is a continuous linear positive (i.e., sending nonnegative functions to nonnegative functions) map $U$ with $\|U\| \leq 1$ and $U^* = U$.

Definition (see [1]). A polymorphism $\Pi$ of a Lebesgue–Rokhlin space $(X_0, \mu_0)$ to itself is a diagram

$$(X_0, \mu_0) \xrightarrow{\pi_1} (X_0 \times X_0, \mu) \xrightarrow{\pi_2} (X_0, \mu_0)$$

consisting of an ordered triple of Lebesgue spaces. Here $\pi_1$ and $\pi_2$ are the projections to the first and second components of the product space $(X_0 \times X_0, \mu)$, and the measure $\mu$ satisfies the conditions $\pi_1 \mu = \pi_2 \mu = \mu_0$. It is obvious that the bistochastic measure $\mu$ with fixed projections uniquely determines the polymorphism.

Consider a polymorphism $\Pi$ of a Lebesgue–Rokhlin space $(X_0, \mu_0)$ to itself. Let

$$T_1 : L^2_{\mu_0}(X_0) \rightarrow L^2_\mu(X_0 \times X_0) \quad \text{and} \quad T_2 : L^2_{\mu_0}(X_0) \rightarrow L^2_\mu(X_0 \times X_0)$$

be the isometric inclusions defined by the formulas $(T_1 f)(x) = f(\pi_1 x)$ and $(T_2 f)(x) = f(\pi_2 x)$. Then the operator $U_{\Pi} : L^2(X_0) \rightarrow L^2(X_0)$ defined by $U_{\Pi} = T_1^* T_2$ is a Markov operator. Conversely, for any Markov operator $U : L^2(X_0) \rightarrow L^2(X_0)$ there exists a unique polymorphism $\Pi$ such that $U = U_{\Pi}$ (see [2]). That is, there is a one-to-one correspondence between polymorphisms and Markov operators.

Let $P_1$ and $P_2$ be the orthogonal projections in $L^2_{\mu_0}(X_0 \times X_0)$ to the subspaces of functions depending on the first and the second coordinate, respectively. Then, obviously, $T_1^* = T_1^{-1} P_1$ and $U_{\Pi} = T_1^{-1} P_1 T_2$. Similarly, $U_{\Pi}^* = T_2^{-1} P_2 T_1$.

We will denote by $\sigma(U)$ the spectrum of an operator $U$.

As shown in Sec. 9 of [2], the problem of classification of polymorphisms up to uniformity can be reduced to the study of $\sigma(P_1 P_2 P_1)$. The following proposition helps to compute this spectrum.

Proposition. If $P_1 P_2 P_1$ is not an identity, then $\sigma(P_1 P_2 P_1) = \sigma((P_1 P_2)\big|_{\text{Im} P_1}) \cup \{0\} = \sigma(U_{\Pi} U_{\Pi}^*) \cup \{0\}$, where $\text{Im} P_1$ denotes the image of $P_1$, that is, the subspace of functions depending on the first coordinate only.

Proof. Since $P_1 P_2 P_1$ is not an identity projection, we have $0 \in \sigma(P_1 P_2 P_1)$. Assume that $\lambda \notin \sigma(P_1 P_2 P_1)$, that is, the operator $R(\lambda) = (P_1 P_2 P_1 - \lambda I_d)^{-1}$ is bounded. Then the operator $R(\lambda)\big|_{\text{Im} P_1} = ((P_1 P_2)\big|_{\text{Im} P_1} - \lambda I_d\big|_{\text{Im} P_1})^{-1}$ is also bounded, $\lambda \notin \sigma((P_1 P_2)\big|_{\text{Im} P_1})$. Hence $\sigma(P_1 P_2 P_1) \supset \sigma((P_1 P_2)\big|_{\text{Im} P_1}) \cup \{0\}$.

On the other hand, let $\lambda \notin \sigma((P_1 P_2)\big|_{\text{Im} P_1})$ and $\lambda \neq 0$. Then the operator $R(\lambda) = ((P_1 P_2)\big|_{\text{Im} P_1} - \lambda I_d\big|_{\text{Im} P_1})^{-1}$ defined on $\text{Im} P_1$ is bounded. This implies that the operator $R(\lambda) = \lambda^{-1} (P_1 P_2 P_1 - \lambda I_d)$ is bounded. Hence, $R(\lambda) P_1 = \lambda^{-1} (P_1 - P_1) P_1 P_2 P_1 - \lambda I_d) P_1 + (\lambda I_d - P_1) = P_1 + (\lambda I_d - P_1) = I_d$ and, similarly, $R(\lambda) P_2 P_1 = \lambda^{-1} (P_2 - P_2) P_1 P_2 P_1 - \lambda I_d) P_2 + (\lambda I_d - P_2) = I_d$, $\lambda \notin \sigma(P_1 P_2 P_1)$, we see that $\sigma(P_1 P_2 P_1) \subset \sigma((P_1 P_2)\big|_{\text{Im} P_1}) \cup \{0\}$ and the first equality from the statement of the proposition is proved.

The second one easily follows from the equalities

$$(P_1 P_2)\big|_{\text{Im} P_1} = (T_1 T_1^{-1} P_1 T_2 T_2^{-1} P_2 T_1 T_1^{-1})\big|_{\text{Im} P_1} = (T_1 U_{\Pi} U_{\Pi}^* T_1^{-1})\big|_{\text{Im} P_1} \times$$

*St. Petersburg State University, St. Petersburg, Russia, e-mail: levin@un Bay ru.

2. Computing the Spectrum

Let \( n \) be a positive integer. By \( \mathbb{T}^n \) we denote the \( n \)-dimensional torus. Consider a linear hyperbolic automorphism \( T: \mathbb{T}^n \to \mathbb{T}^n \) determined by the multiplication by an \( n \times n \) integer-valued matrix with determinant \( \pm 1 \) having no eigenvalues with absolute value 1. Denote by \( A \) the transpose of this matrix. Let \( d \) be the natural metric on the torus, \( N \subset \mathbb{T}^n \) be the subset of points homoclinic to 0 for the automorphism \( T \), that is, the subset of points \( \xi \in \mathbb{T}^n \) such that \( \lim_{|k| \to \infty} d(T^k \xi, 0) = 0 \). It is known that the set \( N \) is countable. Let \( \nu \) be a probability measure on \( N \).

Consider the operator \( U^T : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n) \) given by

\[
U^T \phi(x) = \int_{\mathbb{T}^n} \phi(Tx + \xi) \, d\nu(\xi).
\]

Obviously, \( U^T \) is a Markov operator. Hence there exists a polymorphism \( \Pi \) such that \( U^2 = U^T \). For brevity, in what follows we will denote \( U^T \) by \( U^T \).

The operator \( U^T \) was introduced in [3] as an example of a random perturbation of an algebraic dynamical system. Later, this construction was generalized in [1].

For \( c \in \mathbb{Z}^n \) we denote by \( f_c \in L^2(\mathbb{T}^n) \) the character of the torus given by

\[
f_c(x) = e^{2\pi i \langle c, x \rangle},
\]

where \( \langle c, x \rangle \) denotes the inner product of \( c \) and \( x \). It is known that the functions \( \{ f_c \mid c \in \mathbb{Z}^n \} \) form an orthonormal basis in \( L^2(\mathbb{T}^n) \). By

\[
\hat{\nu}(c) = \int_{\mathbb{T}^n} e^{2\pi i \langle c, \xi \rangle} \, d\nu(\xi)
\]

we denote the Fourier coefficients of the measure \( \nu \), and we set \( C_\nu = \inf_{c \in \mathbb{Z}^n} |\hat{\nu}(c)| \). It is obvious that for all \( c \in \mathbb{Z}^n \) we have \( 0 \leq |\hat{\nu}(c)| \leq 1 \) and, therefore, \( 0 \leq C_\nu \leq 1 \).

**Lemma.** If \( c \) is a fixed element of \( \mathbb{Z}^n \), then \( \hat{\nu}(A^k c) \to 1 \) as \( |k| \to \infty \).

**Proof.** Indeed,

\[
|\hat{\nu}(A^k c) - 1| = \left| \int_{\mathbb{T}^n} e^{2\pi i \langle A^k c, \xi \rangle} \, d\nu(\xi) - 1 \right| = \left| \int_{\mathbb{T}^n} e^{2\pi i \langle c, T^k \xi \rangle} \, d\nu(\xi) - 1 \right|
\]

\[
= \left| \int_{\mathbb{T}^n} (e^{2\pi i \langle c, T^k \xi \rangle} - e^{2\pi i \langle c, 0 \rangle}) \, d\nu(\xi) \right| \leq 4\pi \|c\|_2 \int_{\mathbb{T}^n} d(T^k \xi, 0) \, d\nu(\xi) = 4\pi \|c\|_2 \int_{\mathbb{T}^n} d(\xi, 0) \, dT^k \nu(\xi).
\]

We have used the mean value theorem separately for the real and imaginary parts and the Cauchy-Schwarz inequality. Since \( T^k \nu \overset{w}{\to} \delta_0 \) as \( |k| \to \infty \) (see Lemma 2 from [3]), we obtain the required result.

Let \( D \) be the open unit disk, \( \overline{D} \) its closure, the closed unit disk, and \( \partial D \) be its boundary, the unit circle. Since \( \|U^T\| = 1 \), it is known that \( \sigma(U^T) \subset \overline{D} \).

**Theorem 1.** Assume that \( \hat{\nu}(c) \neq 0 \) for all \( c \in \mathbb{Z}^n \). Then

- \( \partial D \subset \sigma(U^T) \subset \{ \lambda \in \mathbb{C} \mid C_\nu \leq |\lambda| \leq 1 \} \). If \( C_\nu = 0 \), then \( \sigma(U^T) = \overline{D} \).
- The orthogonal complement to the subspace of constants in \( L^2(\mathbb{T}^n) \) can be decomposed into the direct sum of pairwise orthogonal \( U^T \)-invariant subspaces such that the spectrum of \( U^T \) restricted to every such subspace equals \( \partial D \); the operator \( U^T \) has no nonconstant eigenfunctions.

**Proof.** Note that

\[
U^T f_c(x) = \int_{\mathbb{T}^n} e^{2\pi i \langle c, T^k \xi \rangle} \, d\nu(\xi) = \hat{\nu}(c) f_{A^k c}(x).
\]

(1)

Consider the subspaces \( T_c = \langle f_{A^n c} \mid n \in \mathbb{Z} \rangle \) for all \( c \neq 0 \). Obviously, for any \( c_1 \) and \( c_2 \) the subspaces \( T_{c_1} \) and \( T_{c_2} \) either coincide or are orthogonal to each other. Using (1), we see that every subspace \( T_c \) is \( U^T \)-invariant and \( U^T |_{T_c} \) is a weighted shift operator (see, for example, [4]).

188