RAMIFICATION IN ELEMENTARY ABELIAN EXTENSIONS

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The paper is devoted to some properties of ramification invariants in infinite Abelian extensions of exponent \( p \) for a class of complete discrete valuation fields that includes 2-dimensional local fields of prime characteristic \( p \). In particular, it is proved that the maximal such extension with a prescribed upper bound of ramification breaks has finite depth of ramification, and this depth is computed. Bibliography: 5 titles.

In the present paper, ramification in extensions of complete discrete valuation fields \( L/K \) with imperfect residue field is studied. Note that (see the discussion in [5]) to work efficiently with intermediate extensions, to calculate ramifications of composites, and so on, it is insufficient to consider the classical ramification invariants (ramification breaks in upper and lower numberings, the order of the different, and so on). Also, we are facing the problem of complete description of ramification in the extension \( L/K \), which is likely to have a nondiscrete geometric nature.

One approach to studying the ramification in the extension \( L/K \) is to consider various \( LK'/K' \), where the extensions \( K'/K \) are taken from a certain fixed (rather simple) class. So, the classical theory of elimination of higher ramification is based on the fact that \( LK'/K' \) is weakly unramified (has ramification index 1) for a constant extension \( K'/K \) (i.e., for an extension, which is defined over a subfield \( K \) with perfect residue field).

We suggest to investigate the possibility of obtaining a sufficiently complete description of the ramification in \( L/K \), considering the classical ramification invariants in all possible extensions \( LK'/K' \), where \( K'/K \) are finite elementary Abelian \( p \)-extensions (i.e., Abelian extensions of exponent \( p \), which is equal to the characteristic of the residue field). With this aim in mind, in the present paper we study some properties of elementary Abelian extensions of complete discrete valuation fields.

In particular, we prove that the ramification depth of the maximal Abelian extension of exponent \( p \) with a prescribed upper bound of ramification breaks is finite, and this depth is computed (under the assumption that the complete discrete valuation field \( K \) has characteristic \( p \) and \( [\bar{K} : \bar{K}^p] = p \)).

Notation and known facts

Throughout the paper, \( K \) is a complete discrete valuation field of characteristic \( p > 0 \); the notation \( \mathcal{O}_K, m_K, \bar{K}, \) and \( v_K \) is standard (for example, see [5, 1, 3]): \( \pi_K \) is an arbitrary uniformizing element of \( K \) unless otherwise stipulated. For \( a \in \mathcal{O}_K \), by \( \bar{a} \) we denote its residue class in \( \bar{K} \). We assume that \( [\bar{K} : \bar{K}^p] = p \).

For a cyclic extension \( L/K \) of degree \( p \), by \( s(L/K) \) we denote its (logarithmic) ramification break, i.e., the Swan number of the nontrivial automorphism \( \sigma \) of this extension:

\[
s(L/K) = \min_{b \in L^*} v_L(b^p/b - 1).
\]

It is well known that if \( L/K \) is given by the Artin–Schreier equation \( x^p - x = a \), \( v(a) < 0 \), and \( (p, v(a)) = 1 \), then \( L/K \) is wild and \( s(L/K) = -v(a) \). If \( v(a) = -pi, i > 0 \), and \( \pi_K^i a \notin \bar{K}^p \), then \( L/K \) is ferocious and \( s(L/K) = i \).


For a natural number \( i \), we denote by \( K_i \) the composite of all extensions \( L/K \) such that \( L/K \) is given by the equation \( x^p - x = a, a \in \mathfrak{m}_K^i \). If \( G = \text{Gal}(K^{(p)}/K) \), where \( K^{(p)}/K \) is the maximal Abelian extension of \( K \) of exponent \( p \), then, by what has been said in the previous paragraph, \( K_i \) corresponds to the \( i \)th subgroup of the Kato ramification group \( G^i \subset G \).

Next, represent \( K \) in the form \( F((\pi)) \) and, for a natural number \( i \), denote by \( K_{i,F} \) the composite of all extensions \( L/K \) such that \( L/K \) is given by the equation \( x^p - x = a, a \in \mathfrak{m}_K^i + \mathfrak{m}_{F((\pi))}^i \). It is easy to see that \( K_{i,F} \) depends on the choice of \( F \) but does not depend on the choice of \( \pi \); we have \( K_i \subset K_{i,F} \subset K^{p_i} \).

By definition, a \( C \)-extension is an arbitrary infinite extension that is the composite of countably many cyclic extensions of degree \( p \).

1. Auxiliary statements

**Lemma 1.1.** Let \( i \) be a natural number not divisible by \( p \). Then for any extension \( K'/K \) of degree \( p \) such that \( K' \subset K_i, K' \not\subset K_{i-1} \), we have \( K_i \subset K_{i,F} \), where \( F \) is the image in \( K' \) of an arbitrary subfield of representatives in the field \( K \).

**Proof.** Since \( s(K'/K) = i \), one can choose uniformizing elements \( \pi \) and \( \pi_1 \) in \( K \) and \( K' \), respectively, so that \( \pi \equiv \pi_1 \mod \mathfrak{m}_{K_1}^{(p-1)i} \). Hence we conclude that \( \pi j \equiv \pi_1^{pj} \mod \mathfrak{m}_{K_1}^{pj + (p-1)i} \) for all integer \( j \). Consequently,

\[
\mathfrak{m}_K^{i} \subset \mathfrak{m}_{K((\pi_1))}^{i} + \mathfrak{m}_{K'}^{i}.
\]

\( \square \)

The following lemma is an analog of Lemma 3.5 in [3] for a ferocious extension.

**Lemma 1.2.** Let \( K = F((\pi)) \), and let \( L/K \) be given by the equation \( x^p - x = a, a \in \mathfrak{m}_K^i + \mathfrak{m}_{F((\pi))}^j \), where \( i \) and \( j \) are natural numbers such that \( i/p < j \leq i \). Assume that \( \pi^{pj} a \not\equiv K^p \). Then \( F \subset \mathfrak{O}_L^p + \mathfrak{m}_L^{pj-i} \).

**Proof.** Let \( a = (x^p - x)(t^p - x)^{-1} \). Then \( t^p - \pi^{(p-1)j} t = d \), whence

\[
d \in \mathfrak{O}_L^p + \mathfrak{m}_L^{pj - i} \subset \mathfrak{O}_L^p + \mathfrak{m}_L^{pj - i}.
\]

Let \( d' \in F, d' \equiv d \mod \mathfrak{m}_K^i \). From \( a \in \mathfrak{m}_K^i + \mathfrak{m}_{F((\pi))}^j \) it follows that

\[
d \in F((\pi^p)) + \mathfrak{m}_K^{pj - i} = F^{p}[d']((\pi^p)) + \mathfrak{m}_K^{pj - i}.
\]

This easily implies that

\[
d' \in F^{p}[d']((\pi^p)) + \mathfrak{m}_K^{pj - i} \subset \mathfrak{O}_L^p + \mathfrak{m}_L^{pj - i}.
\]

It remains to use the fact that \( F = F^{p}[d'] \) once again. \( \square \)

**Lemma 1.3.** Let \( E/K \) be an extension. Assume that for natural \( i \) we have \( E \subset K_{i,F} \), where \( F \) is an arbitrary subfield of representatives in the field \( K \), and \( E \not\subset K_i \). We take a minimal natural number \( j \) such that \( E \subset K_{pj} \). Then for any extension \( K'/K \) of degree \( p \) such that \( K' \subset E \) and \( s(K'/K) = j \), we have \( K_{i,F} \cap K_{pj} \subset K_i \). (In particular, \( E \subset K_i \).)

**Proof.** Without loss of generality, we may assume that \( E \not\subset K_{i-1,F} \). Then it is clear that \( i/p < j \leq i \). Let \( L/K \) be an arbitrary subextension of \( K_{i,F} \cap K_{pj} \) of degree \( p \); then \( L/K \) can be given by the equation \( x^p - x = a, a \in \mathfrak{m}_K^i + \mathfrak{m}_{F((\pi))}^j \). Lemma 1.2 implies that

\[
\pi^{pj} a \in \mathfrak{O}_{K'}^p + \mathfrak{m}_{K'}^{pj - i},
\]

i.e., \( \pi^{pj} a \equiv b^p \mod \mathfrak{m}_{K'}^{pj - i} \) for some \( b \in \mathfrak{O}_{K'}^p \). The extension \( LK'/K' \) can be obtained by adjoining a root of the equation \( x^p - x = a - \pi^{pj} b^p + \pi^{-j} b \). Since \( a - \pi^{pj} b^p + \pi^{-j} b \in \mathfrak{m}_K^i \), we get \( LK' \subset K_i \), whence \( E \subset K_i \) by the arbitrariness of the choice of \( L/K \). \( \square \)