ABSTRACT. The article is devoted to the investigation of semirings with idempotent multiplication. General structure theorems for such semirings are proved. We focus on the study of the class $\mathfrak{M}$ of all commutative multiplicatively idempotent semirings. We obtain necessary conditions under which semirings from $\mathfrak{M}$ are subdirectly irreducible. We consider some properties of the variety $\mathfrak{M}$. In particular, we show that $\mathfrak{M}$ is generated by two of its subvarieties, defined by the identities $3x = x$ and $3x = 2x$. We explore the variety $\mathfrak{N}$ generated by two-element commutative multiplicatively idempotent semirings. It is proved that the lattice of all subvarieties of $\mathfrak{N}$ is a 16-element Boolean lattice.

1. Background Information

Some results of this paper were announced in [11,14–17].

A semiring is an algebraic system $(S, +, \cdot)$ with binary operations an addition $+$ and a multiplication $\cdot$ such that $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup, and the multiplication is distributive over the addition.

A semigroup is called a semilattice if it is idempotent and commutative.

An element $\theta$ of any semirings $S$ is called an absorbing element for multiplication (or multiplicatively absorbing) if $x \cdot \theta = \theta \cdot x = \theta$ for all $x \in S$ (respectively, $x + \theta = \theta$). An element of a semiring is called absorbing if it is multiplicatively absorbing and additive absorbing. A semiring $S$ is called a semiring with zero 0 if there exists an element 0 that is neutral for addition and multiplicative absorbing. A semiring $S$ is called a semiring with unit 1 if there exists an element 1 that is neutral for multiplication.

Note that for any semiring $S$ the zero element 0 or the absorbing element $\infty$ may be attached to $S$ in a natural way. Let us denote these semirings by $S \cup \{0\}$ and $S \cup \{\infty\}$, respectively.

A semiring $S$ is called commutative if $xy = yx$ for all $x, y \in S$. A semiring $S$ is called multiplicatively idempotent (additively idempotent if $xx = x$ (respectively, $x + x = x$) for all $x \in S$. A semiring $S$ is called idempotent if it is multiplicatively idempotent and additively idempotent. A semiring $S$ is called distributive if there is a dual distributive law: $x + yz = (x + y)(x + z)$ for all $x, y, z \in S$. An idempotent semiring with the identity $x + y = xy$ is called a mono-semiring. We say that a semiring $S$ has constant addition if there exists an element $t \in S$ such that $x + y = t$ for all $x, y \in S$ (equivalently, $S$ satisfies the identity $x + y = u + v$). A semiring $S$ is called rectangular if $xyx = x$ for all $x, y \in S$ (by analogy with the corresponding semigroups).

An equivalence relation $\rho$ on any semiring $S$ is called a congruence if

$$x \rho y \implies (x + z) \rho (y + z) \& (xz) \rho (yz) \& (zx) \rho (zy)$$

for all $x, y, z \in S$.

There are two trivial congruences in any semiring $S$:

1. the null congruence $0_S$ (which is the equality relation),
2. the identity congruence $1_S$ (which is the genus-one congruence).
A semiring $S$ is called subdirectly irreducible if on it there exists a least nonzero congruence. A non-singleton semiring is called congruence-simple if there are no nontrivial congruences. Congruence-simple semirings are subdirectly irreducible.

Two classical Birkhoff theorems are the base for studying semirings varieties. Recall that a variety of semirings is the class of all semirings that satisfy a certain set of semirings identities. From the first Birkhoff [2, p. 185] theorem the following result follows: any class of semirings is a variety if and only if it is closed under taking subsemirings and homomorphic images and any direct products. By the the second Birkhoff theorem [2, p. 115], any semiring is a subdirect product of subdirectly irreducible semirings.

Suppose that $I$ is any ideal semiring $S$ and $\rho(I)$ is a congruence such that

$$x \rho(I) y \iff \exists s, t \in I \ (x + s = y + t).$$

Then the congruence $\rho(I)$ is called a Bourne congruence.

A proper ideal $I$ of a semiring $S$ is called subtractive (strong) if the fact that $a, a+b \in I$ (respectively, $a+b \in I$) implies the fact that $b \in I$. A proper ideal $I$ of a semiring $S$ is called prime if the fact that $ab \in I$ implies the fact that $a \in I$ or $b \in I$ for all $a, b \in S$. Let us denote by $\text{Spec} S$ the set of all prime ideals of a semiring $S$.

By definition, put

$$na = a + \cdots + a$$

for any element $a$ of a semiring $S$ and any natural number $n \geq 2$. If $S$ is a semiring with unit 1, then we write $na = n$ for $a = 1$. In particular, $1 + 1 = 2$ and $1 + 1 + 1 = 3$.

Let $S$ be a semiring with zero 0, and $r(S)$ be the set of all elements that belong to the semiring $S$ and have additive inverse elements. Obviously, $r(S)$ is a subring and strong ideal of $S$. A semiring $S$ with zero 0 is called a zero-sum free semiring if $r(S) = \{0\}$.

**Proposition 1.1.** Let $S$ be a multiplicatively idempotent semirings. The following conditions hold:

1. $x + xy + yx + y = x + y$ for all $x, y \in S$; in particular, $x + x + x + x = x + x$ (or $4x = 2x$);
2. if $S$ is a semiring with unit 1, then 1 is the unique invertible element in $S$;
3. if $S$ is a semiring with zero 0, then $x + y = 0 \implies x = y$ for all $x, y \in S$;
4. $Sa = bS \implies a = b$ for all $a, b \in S$;
5. if either $aS = bS$ or $Sa = Sb$, then $a = b$ for all $a, b \in S$.

**Proof.** (1). We have $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$ for all $x, y \in S$. For $x = y$ we have $x + x = x + x + x + x$.

(2). Assume that $ab = 1$ for some $a, b \in S$. Then $a = a \cdot 1 = a \cdot ab = a^2b = ab = 1$.

(3). Since for a multiplicatively idempotent semiring $S$ the set $r(S)$ is a Boolean subring, we have (3).

(4). We have $a = bs$ and $ta = b$ for some $s, t \in S$. Hence, $a = bs = b^2s = ba = ta^2 = ta = b$.

(5). Since $aS = bS$, we have $a = bs$ and $b = at$ for some $s, t \in S$. Hence, $a = ba$ and $b = ab$. Since $Sa = Sb$, we have $a = ab$ and $b = ba$. Hence, $a = b$.

**Corollary 1.1.** The factor semiring $S/\rho(2S)$ is a Boolean ring.

**Corollary 1.2.** Any principal ideal of a commutative multiplicatively idempotent semiring is generated by a uniquely determined element.

**Example 1.1.** We define a multiplication operation $\cdot$ on an (upper) semilattice $(S, +)$ by the identity $xy = x$. We get the idempotent semiring $(S, +, \cdot)$ such that $Sx = S$ for all $x \in S$. If $S$ contains more than one element, then the fact that $Sa = Sb$ does not imply the fact that $a = b$ for $a \neq b$ in $S$. If we define a multiplication on a semilattice $(S, +)$ by the identity $xy = y$, then we get an idempotent semiring $S$ such that $aS = S$ for all $a \in S$.

For any semiring $S$, denote by $\leq$ the “difference” relation such that

$$x \leq y \iff x = y \text{ or } \exists z \in S \ (x + z = y).$$