We study the stability of regular potential critical points of the energy functional of a two-phase elastic medium. We prove that such a point minimizes the energy functional.

Bibliography: 5 titles.

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1 Introduction

The strain energy functional of a two-phase elastic medium is defined by

\[ I[u, \chi, t] = \int_{\Omega} \left\{ \chi(F^+(\nabla u) + t) + (1 - \chi)F^-(\nabla u) \right\} dx, \tag{1.1} \]

where \( \Omega \subset \mathbb{R}^m, m \geq 2, \) is a bounded domain, \( t \) is the temperature, the characteristic function \( \chi(x) \) determines distribution of phases in \( \Omega, \) the displacement field \( u = u(x) \) is an \( m \)-dimensional vector-valued function, the matrix \( \nabla u \) consists of the coefficients \( u^i_{xj}, \) and the quadratic energy densities \( F^{\pm}(M) \) (here \( M \in \mathbb{R}^{m \times m} \) is the space of \( m \times m \)-matrices) has the form

\[ F^{\pm}(M) = a^{\pm}_{ijkl}(e(M) - \zeta^{\pm})_{ij}(e(M) - \zeta^{\pm})_{kl}. \tag{1.2} \]

The coefficients \( a^{\pm}_{ijkl} \) satisfy the traditional symmetry and positive definiteness conditions, the residual strain tensors are given by symmetric matrices \( \zeta^{\pm}, \) and the matrix

\[ e_{ij}(M) = \frac{M_{ij} + M_{ji}}{2}, \quad M \in \mathbb{R}^{m \times m} \tag{1.3} \]

for \( M = \nabla u \) coincides with the strain tensor. We assume summation with respect to repeated indices from 1 to \( m. \)
For the domain of the functional (1.1) we take the set of admissible displacement fields \( \mathcal{X} \) and admissible phase distributions \( Z \)

\[ \mathcal{X} = W^{1,2}(\Omega, \mathbb{R}^m), \quad Z = \{ \text{all measurable characteristic functions in } \Omega \}. \]  

By an \textit{equilibrium state} for fixed \( t \) we mean a solution \( \tilde{u}_t, \tilde{\chi}_t \) to the variational problem

\[ I[\tilde{u}_t, \tilde{\chi}_t, t] = \inf_{u \in \mathcal{X}, \chi \in Z} I[u, \chi, t], \quad \tilde{u}_t \in \mathcal{X}, \quad \tilde{\chi}_t \in Z. \]  

A pair \( \tilde{u}_t, \tilde{\chi}_t \) is called a \textit{critical point} of the functional (1.1) for fixed \( t \) if

\[ \int_{\Omega} \left\{ \tilde{\chi}_t F_{M_{ij}}^+ (\nabla \tilde{u}_t) + (1 - \tilde{\chi}_t) F_{M_{ij}}^- (\nabla \tilde{u}_t) \right\} v_{x_j}^i \, dx = 0 \quad \forall v \in \mathcal{X}, \]  

\[ \int_{\Omega} \left\{ \tilde{\chi}_t ((F^+ (\nabla \tilde{\chi}_t) + t) \delta_{kj} - \tilde{u}_{tx_k}^i F_{M_{ij}}^+ (\nabla \tilde{u}_t)) \right\} + (1 - \tilde{\chi}_t)(F^- (\nabla \tilde{\chi}_t) \delta_{kj} - \tilde{u}_{tx_k}^i F_{M_{ij}}^- (\nabla \tilde{u}_t)) \right\} h_{x_j}^k \, dx = 0 \quad \forall h \in C_0^1(\Omega, \mathbb{R}^m) \]  

(the subscript of \( F^\pm \) means the derivative with respect to the entries of the matrix \( M \)). The left-hand side of (1.6) is obtained by varying the functional (1.1) with respect to the first variable, and the left-hand side of (1.7) by varying (1.1) with respect to the second variable (cf. [1]). Therefore, the identities (1.6) and (1.7) are necessary conditions for extremum of the functional (1.1). Consequently, an equilibrium state (if exists) is a critical point of the energy functional.

A critical point \( \tilde{u}_t, \tilde{\chi}_t \) is \textit{regular} if there exists an open set \( \omega \subset \Omega, \emptyset \neq \omega \neq \Omega \), such that \( \partial \omega \cap \Omega \) consists of a finite collection of \( m - 1 \)-dimensional surfaces \( \Gamma_l, l = 1, \ldots, L \),

\[ \tilde{\chi}_t(x) = \begin{cases} 1, & x \in \omega, \\ 0, & x \in \Omega \setminus \omega, \end{cases} \quad \tilde{u}_t \in C^2(\omega, \mathbb{R}^m) \cap C^2(\Omega \setminus \omega, \mathbb{R}^m) \cap C(\overline{\Omega}, \mathbb{R}^m). \]  

For a regular critical point \( \tilde{u}_t, \tilde{\chi}_t \) the relations (1.6) and (1.7) are equivalent to the relations

\[ \left\{ \begin{array}{l} - \frac{d}{dx_j} F_{M_{ij}}^+ (\nabla \tilde{u}_t(x)) = 0, \quad x \in \omega, \\ - \frac{d}{dx_j} F_{M_{ij}}^- (\nabla \tilde{u}_t(x)) = 0, \quad x \in \Omega \setminus \omega, \\ [F_{M_{ij}}(\nabla \tilde{u}_t)]_{\Gamma_l n_j} = 0, \quad i = 1, \ldots, m, \quad l = 1, \ldots, L, \\ [\Phi_{M_{ij}}(\nabla \tilde{u}_t)]_{\Gamma_l n_i n_j + t} = 0, \quad l = 1, \ldots, L, \\ \Phi_{M_{ij}} (M) = F^\pm (M) \delta_{ij} - M_{kl} F_{M_{kl}}^\pm (M), \quad l = 1, \ldots, L, \end{array} \right. \]  

respectively. Here, \( n = (n_1, \ldots, n_m) \) is the unit outward normal to \( \partial \omega \cap \Omega \) and \([\cdot]_{\Gamma_l}\) means the jump through the surface \( \Gamma_l \). We emphasize that for computing the jump we from the limit value of a function on \( \partial \omega \cap \Omega \) from the side of \( \omega \) (where \( F^+ \) is defined) we subtract the limit value on \( \partial \omega \cap \Omega \) from the side of \( \Omega \setminus \omega \) (where \( F^- \) is defined).

In elasticity theory, \( F_{M_{ij}}^\pm \) are called the \textit{stress tensors} and \( \Phi_{ij}^\pm \) are referred to as the \textit{chemical potential tensors}. The conditions (1.9) are classical conditions for equilibrium of a composite