We study multiple sampling, interpolation, and uniqueness for the classical Fock space in the case of unbounded multiplicities. Bibliography: 6 titles.

Sampling and interpolating sequences in Fock spaces were characterized by Seip and Seip–Wallstén in [3, 4] by means of a certain Beurling-type asymptotic uniform density. The case of uniformly bounded multiplicities was considered by Brekke and Seip [1] who gave a complete description in this situation. Their conditions show that a sequence cannot simultaneously be sampling and interpolating.

Brekke and Seip also asked whether there exist sequences that are simultaneously sampling and interpolating when the multiplicities are unbounded.

In this research note, we formulate some conditions (of geometric nature) for sampling and interpolation. These conditions show that the answer to this question is negative when the multiplicities tend to infinity.

A detailed version of this work will be published elsewhere.

We now introduce the necessary notation. For $\alpha > 0$, define the Fock space $F_\alpha^2$ by

$$F_\alpha^2 = \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_2^2 = \|f\|_{\alpha,2}^2 := \frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} \, dm(z) < \infty \right\}.$$ 

The space $F_\alpha^2$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-\alpha|z|^2} \, dm(z).$$

The sequence

$$e_k(z) = \frac{\sqrt{\alpha^k}}{\sqrt{k!}} z^k, \quad k \geq 0,$$

defines an orthonormal basis in $F_\alpha^2$.

Recall that the translations

$$T_z f(\zeta) = T_\alpha^z f(\zeta) := e^{\alpha z \bar{\zeta} - \frac{\alpha}{2} |z|^2} f(\zeta - z), \quad f \in F_\alpha^2,$$

act isometrically in $F_\alpha^2$.

Let us now define sampling and interpolation in the unbounded multiplicity case. Consider a divisor $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$, where $\Lambda$ is a sequence of points in $\mathbb{C}$ and $m_\lambda \in \mathbb{N}$ is the multiplicity associated with $\lambda$.

The divisor $X$ is called

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**sampling** for $\mathcal{F}_2^2$ if

$$
\|f\|_2^2 \times \sum_{\lambda \in \Lambda} \sum_{k=0}^{m_\lambda-1} |\langle f, T_\lambda e_k \rangle|^2, \quad f \in \mathcal{F}_2^2;
$$

**interpolating** for $\mathcal{F}_2^2$ if for every sequence $v = \{v_\lambda^{(k)}\}_{\lambda \in \Lambda, 0 \leq k < m_\lambda}$ such that

$$
\|v\|_2^2 := \sum_{\lambda \in \Lambda} \sum_{k=0}^{m_\lambda-1} |v_\lambda^{(k)}|^2 < \infty
$$

there exists a function $f \in \mathcal{F}_2^2$ satisfying

$$
\langle f, T_\lambda e_k \rangle = v_\lambda^{(k)}, \quad 0 \leq k < m_\lambda, \quad \lambda \in \Lambda.
$$

As in the situation of classical interpolation and sampling, the separation between points in $\Lambda$ plays an important role. Denote by $D(z, r)$ the disk of radius $r$ centered at $z$.

A divisor $X = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda}$ is said to satisfy the finite overlap condition if

$$
\sup_{z \in \mathbb{C}} \sum_{\lambda \in \Lambda} \chi_{D(\lambda, \sqrt{m_\lambda/\alpha} + C)}(z) < \infty.
$$

If $\Lambda$ is a finite union of $\Lambda_j$ such that the disks $D(\lambda, \sqrt{m_\lambda/\alpha})$, $\lambda \in \Lambda_j$, are disjoint for every $j$, then $X$ satisfies the finite overlap condition.

The following result gives geometric conditions for sampling in the case of unbounded multiplicities.

**Theorem 1.** (a) If $X = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda}$ is a sampling divisor for $\mathcal{F}_2^2$, then $X$ satisfies the finite overlap condition and there exists $C > 0$ such that

$$
\bigcup_{\lambda \in \Lambda} D(\lambda, \sqrt{m_\lambda/\alpha} + C) = \mathbb{C}.
$$

(b) Conversely, if $X = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda}$ satisfies the finite overlap condition and for every $C > 0$ there is a compact subset $K$ of $\mathbb{C}$ such that

$$
\bigcup_{\lambda \in \Lambda: m_\lambda > \alpha C^2} D(\lambda, \sqrt{m_\lambda/\alpha} - C) = \mathbb{C} \setminus K,
$$

then $X$ is a sampling divisor for $\mathcal{F}_2^2$.

**Remark.** Let $X = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda}$ satisfy the conditions of Theorem 1 (b). Then we can find a subset $\Lambda_1 \subset \Lambda$ such that for every $C > 0$ there is a compact subset $K$ of $\mathbb{C}$ satisfying

$$
\bigcup_{\lambda \in \Lambda_1: m_\lambda > \alpha C^2} D(\lambda, \sqrt{m_\lambda/\alpha} - C) = \mathbb{C} \setminus K \quad \text{and} \quad \lim_{\lambda \in \Lambda_1, |\lambda| \to \infty} m_\lambda = +\infty.
$$

The following result gives geometric conditions for interpolation in the case of unbounded multiplicities.

**Theorem 2.** (a) If $X = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda}$ is an interpolating divisor for $\mathcal{F}_2^2$, then there exists $C > 0$ such that the disks $\{ D(\lambda, \sqrt{m_\lambda/\alpha} - C) \}_{\lambda \in \Lambda, m_\lambda > \alpha C^2}$ are pairwise disjoint.

(b) Conversely, if the disks $\{ D(\lambda, \sqrt{m_\lambda/\alpha} + C) \}_{\lambda \in \Lambda}$ are pairwise disjoint for some $C > 0$, then $X$ is an interpolating divisor for $\mathcal{F}_2^2$. 

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