The paper considers strong approximation of continuous functions by positive operators. Estimates in terms of the modulus of continuity and its convex majorant are established. Bibliography: 4 titles.

**Introduction**

Let $C(\mathbb{R})$ be the set of functions $f$ uniformly continuous on $\mathbb{R}$,

$$
\|f\| = \sup_{x \in \mathbb{R}} |f(x)|, \quad \omega(f, h) = \sup_{|t| \leq h} \|f(\cdot + t) - f(\cdot)\|,
$$

$$
J_\sigma(f, x) = \int_{\mathbb{R}} f \left( x + \frac{t}{\sigma} \right) K(t) \, dt.
$$

In [1], the following result was established.

**Theorem A.** Let functions $g, f, \varphi$, and $K$ satisfy the following assumptions: $\varphi'' \in C(\mathbb{R})$, $\|\varphi''\| < +\infty$, and the domain of the function $f$ contains $\varphi(\mathbb{R})$. Assume that $\omega(f, h) \leq \omega(h)$, where $\omega$ is the continuity modulus convex upward, $g = f \circ \varphi$, the kernel $K(t)$ is nonnegative for all $t$ and integrable on $\mathbb{R}$,

$$
\int_{\mathbb{R}} K(t) \, dt = 1, \quad \int_{\mathbb{R}} t^2 K(t) \, dt < +\infty.
$$

Then, for any $y \in \mathbb{R}$ and $\sigma > 0$,

$$
|g(y) - J_\sigma(g, y)| \leq \omega \left( \frac{\|\varphi'(y)\|}{\sigma} \int_{\mathbb{R}} |t| K(t) \, dt + \frac{\|\varphi''\|}{2\sigma^2} \int_{\mathbb{R}} t^2 K(t) \, dt \right).
$$

Theorem A was applied in the following situation: $f$ was continuous on $[-1, 1]$, $\varphi(y) = \cos y$, $K$ was the Boman–Korovkin kernel,

$$
K(t) = 4\pi \frac{\cos \frac{t}{\sigma}}{(t^2 - \pi^2)^2},
\sigma = n \in \mathbb{N}.
$$

As a result, for all $y \in [0, \pi]$, we had

$$
|g(y) - J_n(g, y)| \leq \omega \left( \frac{\sin y}{n} \left( 2Si \pi - \frac{4}{\pi} \right) + \frac{\pi^2}{2n^2} \right).
$$

Setting $y = \arccos x$, for $x \in [-1, 1]$ we find

$$
|f(x) - J_n(f(\cos y), \arccos x)| \leq \omega \left( \frac{\sqrt{1 - x^2}}{n} \left( 2Si \pi - \frac{4}{\pi} \right) + \frac{\pi^2}{2n^2} \right).
$$

It should be emphasized that $J_n(f(\cos y), \arccos x)$ is an algebraic polynomial of degree not exceeding $n - 1$.  

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It was noted that in the case where \( \varphi(y) = \cos y \), the theorem assertion can be somewhat refined. In this connection, inequality (1) can be replaced by the estimate
\[
|f(x) - J_n(f(\cos y), \arccos x)| \leq \omega \left( \frac{\sqrt{1 - x^2}}{n} \left( \frac{2 \sin \pi - \frac{4}{\pi}}{2} \right) + |x| \frac{\pi^2}{2n^2} \right),
\]
where \( x \in [-1, 1] \).

In the present paper, we extend Theorem A along several directions.

Similar results are also established for a wide class of summatory positive approximation methods. In particular, with the use of the modified Bernstein polynomials, Zubov’s theorem (see [2, Theorem 3]), stating that functions continuous on the entire axis and having finite limits, as the argument tends to infinity, can be uniformly approximated by certain sums constructed based on distribution functions, was strengthened.

1. Notation. Auxiliary propositions. Main results

1.1. In the sequel, \( \mathbb{R} \), \( \mathbb{R}_+ \), \( \mathbb{Z}_+ \), and \( \mathbb{N} \) are the sets of reals, nonnegative reals, nonnegative integers, and positive integers, respectively. The notation \( k = \overline{a, b} \), where \( a, b \in \mathbb{R}, a \leq b \), means that \( k \) runs over all integers in between \( a \) and \( b \), including \( a \) and \( b \) if they are integers. Functions with a removable singularity at a certain point are defined at this point by continuity; in other cases, the symbol \( 0 \) is understood as \( 0; \mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\} \).

An increasing function \( \omega: \mathbb{R}_+ \to \mathbb{R}_+ \) that satisfies the conditions
(a) \( \omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2) \) if \( t_1, t_2 \geq 0 \) (semiadditivity)
and
(b) \( \lim_{t \to 0^+} \omega(t) = \omega(0) = 0 \)
is called the modulus of continuity. The set of all moduli of continuity is denoted by \( \Omega \).

The set of those moduli of continuity which are convex upward on \( \mathbb{R} \) is denoted by \( \Omega^* \).

By \( C[a,b] \) we denote the space of functions \( f: [a,b] \to \mathbb{R} \) continuous on \([a,b] \) equipped with the norm \( \| f \|_{[a,b]} = \max_{x \in [a,b]} |f(x)| \).

If \( f \in C[a,b] \), then \( \omega(f, h; a, b) = \sup |f(x) - f(y)| \), where the supremum is taken over all \( x, y \in [a,b] \) such that \( |x - y| \leq h \).

We set \( \text{Lip}(1; a, b) = \{ f \in C[a, b] : \omega(f, h; a, b) \leq h \text{ for all } h \geq 0 \} \). In the case where the interval in question is obvious from the context, it is sometimes omitted in the notation.

1.2. We will need the following known results.

**Theorem B** ([3, p. 8]). Let a seminorm \( P \) be given on \( C[a,b] \) and satisfy the following conditions: the values
\[
\alpha = \sup_{f \in C[a,b]} \frac{P(f)}{\|f\|_{[a,b]}} \quad \text{and} \quad \beta = \sup_{f \in \text{Lip}(1;a,b)} P(f)
\]
are finite, and the numbers \( M_0 \) and \( M_1 \) satisfy the inequalities \( M_0 > 0, M_0 \geq \alpha, M_1 \geq \beta \). If, under these assumptions, for a function \( f \in C[a,b] \) and \( h \in \mathbb{R}_+ \), we have \( \omega(f, h; a, b) \leq \omega(h) \), where \( \omega \in \Omega^* \), then
\[
P(f) \leq \frac{M_0}{2} \omega \left( \frac{2M_1}{M_0} \right).
\]

**Theorem C** (see [3, p. 5; 4, p. 69]). For any \( \omega \in \Omega \) there exists a modulus of continuity, \( \omega^* \in \Omega^* \), such that for all nonnegative \( t \) and \( \lambda \),
\[
\omega(\lambda t) \leq \omega^*(\lambda t) \leq (\lambda + 1)\omega(t).
\]

**Remark A** (see [3, p. 6]). If \( f \in C[a, b] \), then \( \omega(f, \cdot; a, b) \in \Omega \).