A Version of the $\mathcal{G}$-conditional Bipolar Theorem in $L^0_0(\mathbb{R}^+; \Omega_1, \mathcal{F}, \mathbb{P})$

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1. INTRODUCTION

Let $(\Omega_1, \mathcal{F}, \mathbb{P})$ be a probability space and denote by $L^0_0(\mathbb{R}^+; \Omega_1, \mathcal{F}, \mathbb{P})$ the space of equivalent classes of $\mathbb{R}^+$-valued random variables on $(\Omega_1, \mathcal{F}, \mathbb{P})$, equipped with the topology of convergence in measure. Although this space fails to be locally convex, it was shown in Ref. 3, that an analogue to the bipolar theorem can be obtained for subsets of $L^0_0(\mathbb{R}^+; \Omega_1, \mathcal{F}, \mathbb{P})$: if we place this space in polarity with itself, the bipolar of a set of non-negative random variables is equal to its closed (in probability), solid, convex hull. This result was extended by Ref. 1 in the multidimensional case, replacing $\mathbb{R}^+$ by a closed convex cone $K$ of $[0, \infty)^d$, and by Ref. 12 who provided a conditional version in the unidimensional case. In this paper, we show that the conditional bipolar theorem of Ref. 12 can be extended to the multidimensional case. Using a decomposition result obtained in Ref. 3 and Ref. 1, we also remove the boundedness assumption of Ref. 12 in the one dimensional case and provide less restrictive assumptions in the multidimensional case. These assumptions are completely removed in the case of polyhedral cones $K$.

KEY WORDS: Bipolar theorem; convex analysis; partial order.

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The classical bipolar theorem (see e.g. Ref. 11) can be obtained for subsets of $L^0(\mathbb{R}^+;\Omega_1,F,P)$: if we place this space in polarity with itself, the scalar produce $\langle X,Y \rangle := E[XY]$ taking possibly infinite values, the bipolar of a set of non-negative random variables is equal to its closed, solid, convex hull.

The formulation of this bipolar result in Ref. 3 was originally motivated by application in mathematical finance. In the theory of financial markets, it is usual to rely on a duality relation between some random variables, interpreted as contingent claims, and Radon–Nikodym derivatives of some absolutely continuous martingale measures. As $L^0(\mathbb{R}^+;\Omega_1,F,P)$ remains unchanged under a passage to an equivalent measure, it turns out to be a natural space to work on (see e.g. Ref. 6 for a description of these duality relations).

This result was then extended to subsets of $L^0(K;\Omega_1,F,P)$ where $K$ is a closed convex cone of $[0,\infty)$. This extension in Ref. 1 was motivated by applications to the theory of financial markets with transaction costs. In such markets, portfolios (resp. contingent claims) are naturally modeled by vector valued processes (resp. random variables), as opposed to real valued processes (resp. random variables). Similarly, Radon–Nikodym derivatives need to be replaced by a family of processes with values in a convex cone $K$ of $[0,\infty)$. It follows that the natural space to work on is $L^0(K;\Omega_1,F,P)$ instead of $L^0(\mathbb{R}^+;\Omega_1,F,P)$ (see Ref. 1 for an application of Theorem 2.1 below in mathematical finance and for references).

Finally, in Ref. 12, a conditional version was proved in the unidimensional case for bounded (in probability) subsets of $L^0(\mathbb{R}^+;\Omega_1,F,P)$. It was a first step in the proof of a filtered bipolar theorem, which also has some application in mathematical finance. In particular, it allows to characterize the set of feasible consumption processes in frictionless financial markets (see Ref. 12 for more details.)

In this paper, we show that the conditional bipolar theorem of Ref. 12 can be easily extended to the multidimensional case. Using a decomposition result obtained in Refs. 3 and 1, we also remove the boundedness assumption of Ref. 12 in the one dimensional case and provide less restrictive assumptions in the multidimensional case. These assumptions are completely removed in the case of polyhedral cones $K$.

The rest of the paper is organized as follows. In Section 2, we introduce the notations, describe the results of Refs. 3, 1 and 12, and state our main result. An example of application is given in Section 3. Proofs are collected in Section 4.

In all this paper, inequalities involving random variables have to be understood in the $P$-a.s. sense.