Phase Transition in Vertex-Reinforced Random Walks on $\mathbb{Z}$ with Non-linear Reinforcement

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Vertex-reinforced random walk is a random process which visits a site with probability proportional to the weight $w_k$ of the number $k$ of previous visits. We show that if $w_k \sim k^\alpha$, then there is a large time $T_0$ such that after $T_0$ the walk visits 2, 5, or $\infty$ sites when $\alpha < 1$, =1, or $> 1$, respectively. More general results are also proven.

KEY WORDS: Vertex-reinforced random walks; urn models; Rubin’s construction.

SUBJECT CLASSIFICATION: 60G20; secondary 60K35.

1. INTRODUCTION

Consider nearest-neighbor stochastic process $X_n$, $n = 0, 1, 2, \ldots$, on $\mathbb{Z}$ with transition probabilities

$$P(X_{n+1} = m + 1 | X_n = m) = \frac{w_{L(n,m+1)}}{w_{L(n,m-1)} + w_{L(n,m+1)}},$$

$$P(X_{n+1} = m - 1 | X_n = m) = \frac{w_{L(n,m-1)}}{w_{L(n,m-1)} + w_{L(n,m+1)}},$$

where $L(n,m) := \sum_{i=1}^{n} I\{X_i = m\}$ is the number of visits to the site $m$ by time $n$, and $w_k$, $k \in \mathbb{Z}_+$, is a fixed sequence of positive numbers, referred to as “weights.” Note that the most commonly studied case is when $w_k = k + 1$. 

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In the above form, this process, called Vertex-reinforced random walk, or VRRW for short, has been introduced in Ref. 7, while the notion of the VRRW dates back to Ref. 6 and makes the contrast to (Edge) reinforced random walks defined in Copper smith and Diaconis. Especially after the publication of Ref. 7, VRRW has been drawing a lot of attention. In Ref. 11, VRRW on arbitrary graphs has been studied; in Ref. 1 and 2 some properties of more general nonhomogeneous VRRW on \( \mathbb{Z} \) were investigated. Finally, 10 proves an important conjecture from Pemantle and Volkov (1987, Unpublished manuscript) about the behavior of linearly-reinforced random walk, while Ref. 8 and references therein provide a broad review of various reinforced processes.

Now let \( R = \{ m \in \mathbb{Z} : X_n = m \text{ for some } n \} \) be the range and \( R' = \{ m \in \mathbb{Z} : X_n = m \text{ for infinitely many } n \} \) be the effective range of the VRRW. We say that the VRRW gets stuck if \( R \) is finite. It is also obvious that \( R' \subseteq R \), and that if \( R' \) is not empty, then it must consist of a sequence of consecutive integers.

Throughout the paper, we will write \( w_k \sim k^\alpha \) whenever there exists

\[
0 < \lim_{k \to \infty} \frac{w_k}{k^\alpha} < \infty.
\]

**Theorem 1.** Suppose that \( w_k \sim k^\alpha \). Then

(a) if \( \alpha < 1 \) then \( |R| = \infty \) and \( |R'| \in \{0, \infty\} \);
(b) if \( \alpha = 1 \) and \( w_k \equiv k + 1 \) then \( |R| < \infty \) and \( |R'| = 5 \);
(c) if \( \alpha > 1 \) then \( |R| < \infty \) and \( |R'| = 2 \).

Further we will study three special cases.

2. SUBCRITICAL CASE

Throughout this section we assume that the sequence of weights \( w_k \) satisfies the following conditions:

(a) there exists \( 0 < \gamma \leq 1 \) such that

\[
w_n \geq \gamma w_k \text{ whenever } n > k\tag{2.1}
\]

(\( \gamma = 1 \) corresponds to increasing sequences);

(b) for any \( r > 1 \)

\[
\limsup_{n \to \infty} \frac{w_{\lfloor rn \rfloor}}{w_n} < \infty; \tag{2.2}
\]