Berry–Esseen Bounds for Standardized Subordinators via Moduli of Smoothness

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Abstract We introduce moduli of smoothness techniques to deal with Berry–Esseen bounds, and illustrate them by considering standardized subordinators with finite variance. Instead of the classical Berry–Esseen smoothing inequality, we give an easy inequality involving the second modulus. Under finite third moment assumptions, such an inequality provides the main term of the approximation with small constants, even asymptotically sharp constants in the lattice case. Under infinite third moment assumptions, we show that the optimal rate of convergence can be simply written in terms of the first modulus of smoothness of an appropriate function, depending on the characteristic random variable of the subordinator. The preceding results are extended to standardized Lévy processes with finite variance.

Keywords Berry–Esseen bounds · Subordinator · Lévy process · Moduli of smoothness · Sharp constants · Concentration function

1 Introduction

Let us introduce some notation. We consider complex–valued measurable functions \( f \) defined on \( \mathbb{R} \), the supremum–norm of which is denoted by \( \| f \| \). Let \( m = 1, 2, \ldots \). The usual \( m \)th modulus of smoothness of \( f \) at length \( \varepsilon \geq 0 \) is defined by

\[
    w_m(f; \varepsilon) := \sup\{ |\Delta_h^m f(x)| : x \in \mathbb{R}, 0 \leq h \leq \varepsilon \},
\]

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where $\Delta_h^m f$ stands for the $m$th symmetric difference of $f$ with step $h \geq 0$, that is,

$$
\Delta_h^m f(x) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} f \left( x + \left( \frac{m}{2} - k \right) h \right), \quad x \in \mathbb{R}.
$$

It is well known (cf. Ditzian and Totik [9, p. 37]) that

$$
w_m(f; s \varepsilon) \leq \lceil s \rceil^m w_m(f; \varepsilon), \quad s, \varepsilon \geq 0,
$$

where $\lceil s \rceil$ is the ceiling of $s$, that is, the smallest integer not less than $s$. Also, if $f$ is $m$ times differentiable, then

$$
w_m(f; \varepsilon) \leq \| f^{(m)} \| \varepsilon^m, \quad \varepsilon \geq 0.
$$

Finally, let $Y$ be a symmetric random variable. For any $\varepsilon \geq 0$, we consider the positive linear operator $P_{\varepsilon}$ acting on bounded functions $f$ and defined as

$$
P_{\varepsilon} f(x) := E f(x + \varepsilon Y), \quad x \in \mathbb{R}.
$$

Moduli of smoothness are widely used tools in approximation theory to measure the speed of convergence. In this sense, weighted Ditzian–Totik moduli of smoothness of second order characterize the rate of convergence in approximating a function $f$ by a sequence $(P_n f)_{n \geq 1}$, $P_n$ being a positive linear operator (see, for instance, Ditzian and Totik [9] and Ditzian and Ivanov [8]). Specifically, for operators of the form (1.3) and whenever $Y$ satisfies good moment properties, it can be shown (cf. Adell and Sangüesa [3]) that

$$
c_1 w_2(f; \varepsilon) \leq \| P_{\varepsilon} f - f \| \leq c_2 w_2(f; \varepsilon), \quad \varepsilon \geq 0,
$$

where $c_1$ and $c_2$ are absolute positive constants.

In this paper, we shall be mainly interested in moduli of smoothness of first and second order. In our approach to find rates of convergence in the central limit theorem, we use the following easy lemma, in substitution of the classical Berry–Esseen smoothing inequality. Other smoothing inequalities can be found in Arak and Zaitsev [4, pp. 61–62] and Bentkus et al. [5, Lemma 10].

**Lemma 1.1** Let $f$ and $g$ be bounded functions. For any $\varepsilon \geq 0$, we have

$$
\| f - g \| \leq \| P_{\varepsilon} f - P_{\varepsilon} g \| + \frac{1}{2} E w_2(f; \varepsilon | Y|) + \frac{1}{2} E w_2(g; \varepsilon | Y|).
$$

The proof of this lemma is immediate by taking into account the triangular inequality and the fact that the symmetry of $Y$ implies that

$$
P_{\varepsilon} f(x) - f(x) = \frac{1}{2} E \left( f(x + \varepsilon Y) - 2 f(x) + f(x - \varepsilon Y) \right), \quad \varepsilon \geq 0, \quad x \in \mathbb{R}.
$$

Suppose now that $(F_t)_{t \geq 1}$ is a family of distribution functions converging to the standard normal distribution function $\Phi$ as $t \to \infty$. Choose a square integrable random variable $Y$ whose characteristic function vanishes outside $[-1, 1]$. Applying