Strong Limit Theorems for Weighted Sums of Negatively Associated Random Variables

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Abstract In this paper, we establish strong laws for weighted sums of identically distributed negatively associated random variables. Markinikiewicz-Zygmund’s strong law of large numbers is extended to weighted sums of negatively associated random variables. Furthermore, we investigate various limit properties of Cesàro’s and Riesz’s sums of negatively associated random variables. Some of the results in the i.i.d. setting, such as those in Jajte (Ann. Probab. 31(1), 409–412, 2003), Bai and Cheng (Stat. Probab. Lett. 46, 105–112, 2000), Li et al. (J. Theor. Probab. 8, 49–76, 1995) and Gut (Probab. Theory Relat. Fields 97, 169–178, 1993) are also improved and extended to the negatively associated setting.

Keywords Strong law · Weighted sum · Cesàro mean · Complete convergence · Negatively associated random variable

Mathematics Subject Classification (2000) 60F15

1 Introduction

Let \( \{X, X_n, n \geq 1\} \) be a sequence of identically distributed random variables (r.v.) with \( EX = 0 \). Let \( k_n \leq M n \) (where \( M \) is an integer not depending on \( n \)) be a sequence of positive integers and \( \{a_{ni}, 1 \leq i \leq k_n\} \) be an array of real numbers. Define
a weighted sum by

\[ S_{k_n} = \sum_{i=1}^{k_n} a_{ni} X_i. \]

Note that many useful linear statistics are in this form, e.g., least-squares estimators, nonparametric regression function estimators and jackknife estimates among others. Studies on strong laws for \( S_{k_n} \) are of great interest not only in probability theory but also in statistics.

1.1 Some Review for Independent r.v.’s

There has been much literature on the limit properties of weighted sums when \( \{X, X_n, n \geq 1\} \) are assumed to be independent r.v.’s. For instance, Thrum [39] extended work by Chow [9], and obtained the following result:

**Theorem 1.1** (Thrum [39]) If \( \sum_{i=1}^{k_n} a_{ni}^2 = 1, \) and \( E|X|^p < \infty \) for some \( p \geq 2, \) then \( S_{k_n}/n^{1/p} \to 0 \) a.s.

This result was further extended by Li et al. [25]:

**Theorem 1.2** (Li et al. [25]) If \( E|X|^p < \infty \) (\( p \geq 2 \)), \( \sup_{n,k}|a_{nk}| < \infty \) and \( \sum_{i=1}^{k_n} a_{ni}^2 = O(n^\delta) \) (\( 0 < \delta < 2/p \)), then \( S_{k_n}/n^{1/p} \to 0 \) a.s.

As another example, Bai et al. [4] and Bai and Cheng [3] extended the work of Cuzick [12] and obtained:

**Theorem 1.3** (Bai et al. [4] and Bai and Cheng [3]) Let \( 1/p = 1/\alpha + 1/\beta \) for \( 1 < \alpha, \beta < \infty, \) and \( E|X|^\beta < \infty. \) Suppose

\[
A_{\alpha} := \limsup_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} |a_{ni}|^{\alpha} \right)^{1/\alpha} < \infty. \quad (1.1)
\]

Then the following results hold:

1. If \( 1 < p < 2, \) then \( S_{k_n}/n^{1/p} \to 0 \) a.s.
2. If \( p = 2, \) then \( \limsup_{n \to \infty} |S_{k_n}|/(n \log n)^{1/2} \leq \sqrt{2} A_{2}(E|X|^2)^{1/2} \) a.s.

In addition, Jajte [19] studied the strong law of large numbers (SLLN) for a large class of means for independent and identically distributed (i.i.d.) random variables.

**Theorem 1.4** (Jajte [19]) Let \( g(\cdot) \) be a positive, increasing function and \( h(\cdot) \) a positive function such that \( \phi(y) \equiv g(y)h(y) \) satisfies the following conditions:

(i) For some \( d \geq 0, \phi(\cdot) \) is strictly increasing on \( [d, \infty) \) with range \( [0, \infty). \)
(ii) There exist \( C \) and a positive integer \( k_0 \) such that \( \phi(y + 1)/\phi(y) \leq C, \ y \geq k_0. \)
(iii) There exist constants \( a \) and \( b \) such that \( \phi^2(s) \int_s^{\infty} \frac{1}{\phi^2(x)} dx \leq as + b, \ s > d. \)