ABSTRACT. This paper obtains the weak completeness and decidability results for standard systems of modal logic using models built from formulas themselves. This line of work began with Fine (Notre Dame J. Form. Log. 16:229–237, 1975). There are two ways in which our work advances on that paper: First, the definition of our models is mainly based on the relation Kozen and Parikh used in their proof of the completeness of PDL, see (Theor. Comp. Sci. 113–118, 1981). The point is to develop a general model-construction method based on this definition. We do this and thereby obtain the completeness of most of the standard modal systems, and in addition apply the method to some other systems of interest. None of the results use filtration, but in our final section we explore the connection.

KEY WORDS: canonical formula, modal logic, model-construction method, weak completeness

1. INTRODUCTION

The normal forms of propositional modal logic have been discovered several times. These are the analog of the Scott sentences in modal logic, and they also are generalizations of state descriptions from propositional logic. We’ll define them in due course, but here are some examples:

\[
\begin{align*}
\alpha &= \neg p \land \neg q \land \Diamond (p \land q) \land \Diamond (\neg p \land q) \land \Box (((p \land q)) \lor (\neg p \land q)) \\
\beta &= p \land \neg q \land \Diamond (\neg p \land q) \land \Diamond (\neg p \land q) \land \Box (((\neg p \land q)) \lor (\neg p \land q)) \\
\chi &= \neg p \land q \land \Diamond \alpha \land \Diamond \beta \land \Box (\alpha \lor \beta)
\end{align*}
\]

The primary source on the use of normal forms is Kit Fine’s paper 1975 paper “Normal forms in modal logic” [4]. Presumably Fine called them “normal forms” because every modal formula is equivalent to a disjunction of a finite set of them. In a different way, such sentences serve as characterizing sentences (or approximations to such sentences). This means that the bisimulation type of a given model-world pair is an infinitary sentence built in the manner of the examples above; see [1],
Theorem 11.12. This result will not be important to us, and indeed we shall refer to these formulas as canonical formulas in our development.

Fine [4] claim that “Normal forms have been comparatively neglected in the study of modal sentential logic” seems even more cogent 30 years after its publication. The topic is missing from most recent textbooks, and only a handful of papers discuss it. There are several possible reasons for this. First, normal forms give weak completeness and decidability results, and these can be obtained as well via the method of filtration, as first shown by Lemmon and Scott [8]. So one might reasonably ask what the advantage of normal form proofs could be. This is answered by Fine’s claim that normal form methods are more elegant. Indeed, as David Makinson’s review [9] points out, “[Normal forms are] applied with flair and elegance to the modal logics $K$, $T$, $K4$, and a fairly broad class of ‘uniform’ modal logics. In the case of $K$ the construction turns out to be quite simple; in the other cases it is rather intricate.”

And this brings us to the second possible reason for the neglect of normal forms. There has not been an account of what the method consists of that allows us to ask what it can and cannot do. Thus the original applications in [4] seem in retrospect to be ad hoc. To be more specific on this point, Fine’s main construction builds finite Kripke models from the normal forms themselves. The “intricate” constructions boil down to the specification of a particular accessibility relation on a particular set of normal forms of a given (finite) height and over a given (finite) set of atomic propositions. The original definitions of the subset and the relation are indeed special, and it would appear that they must be tailored logic-by-logic.

This paper attempts to re-open the matter of building Kripke models from the formulas of the logic itself. It develops the topic from scratch in Sections 2 and 3 and then turns to new applications. Our re-working of the topic aims to develop it as a method in the sense that we settle on one main construction, the models $C_{h,n}(L)$ introduced in Section 4. This relates to our point just above. Our definitions are arguably simpler and more ‘canonical.’ We have $C_{h,n}(L) = (C_{h,n}(L), \rightarrow, v)$, where

1. $C_{h,n}(L)$ is a certain set of formulas of modal height $\leq h$ built from the first $n$ atomic propositions, all of which are consistent in the logic $L$.
2. $\alpha \rightarrow \beta$ iff $\alpha \land \Box \beta$ is consistent in $L$.
3. $v(p_i) = \{ \alpha : \vdash \alpha \rightarrow p_i \}$.

Point (3) is what one would expect from any model construction where the worlds are formulas. The important point is (2). This definition comes from the Kozen-Parikh [7] proof of the completeness of propositional dynamic logic. That paper was published 6 years after