ABSTRACT. The well-known picture that sequent derivations without cuts and normal derivations “are the same” will be changed. Sequent derivations without maximum cuts (i.e. special cuts which correspond to maximum segments from natural deduction) will be considered. It will be shown that the natural deduction image of a sequent derivation without maximum cuts is a normal derivation, and the sequent image of a normal derivation is a derivation without maximum cuts. The main consequence of that property will be that sequent derivations without maximum cuts and normal derivations “are the same”.

KEY WORDS: cut elimination, normalization

1. INTRODUCTION

In [6] Gentzen introduced a natural deduction system for intuitionistic predicate logic, the system $NJ$, and a system of sequents for intuitionistic predicate logic, the system $LJ$. Moreover, there are several papers and books [1, 3, 5, 6, 8–11, 13, 14] in which natural deduction systems and systems of sequents for some fragments of intuitionistic logic are compared. In almost all of them two characteristic theorems of these systems, the normalization theorem (from systems of natural deduction) and the cut-elimination theorem (from systems of sequents), were connected. It means that the most important connection between these systems is the connection between normal derivations, i.e. derivations without maximum segments (from the systems of natural deduction) and cut-free derivations, i.e. derivations without cuts (from the systems of sequents). In some papers mentioned above (see for example [8, 10, 14]) it was concluded that “the normalization theorem and the cut-elimination theorem are equivalent”. It seems that maximum segments correspond to cuts, and vice versa. However, the connection is the following (see Theorem 3 and Theorem 4 in Section 5 from [14]):

The image of a cut-free derivation is a normal derivation, (*) but

if a normal derivation is the image of a sequent derivation, then that sequent derivation has some cuts which can be eliminated.

This non-symmetrical picture is the result of differences between systems of sequents and natural deduction systems. We can note that each connection between a system of sequents and a natural deduction system consists of two
important steps. The first step is a connection between derivations of these
two kinds of systems, and particularly a connection between cut-free
derivations and normal derivations. In the second step (by using the results
of the first step) a connection between reductions of cut elimination and
reductions of normalization should be made. We will present problems of the
first step of connections between systems of sequents and natural deduction
systems, and some solutions of these problems.

**Problem 1: Connections between natural deduction derivations and
sequent derivations**

In [6] Gentzen defined a transformation from the set of \(NJ\)-derivations,
\(\text{Der}(NJ)\), to the set of \(LJ\)-derivations, \(\text{Der}(LJ)\). That transformation can be
considered as a map from \(\text{Der}(NJ)\) to \(\text{Der}(LJ)\). On the other hand, in [11]
Prawitz defined an interpretation of derivations from \(\text{Der}(LJ)\) by
derivations from \(\text{Der}(NJ)\).

If we want to define maps which connect derivations of Getzen’s
systems \(LJ\) and \(NJ\), then these maps have the following characteristics.

\((LJ \rightarrow NJ)\) Maps from \(\text{Der}(LJ)\) to \(\text{Der}(NJ)\) have the following problems:

\((LJ \rightarrow NJ_1)\) In the system \(NJ\) there are not explicit structural rules:
thinning, contraction and interchange, so these rules from the system \(LJ\)
do not have corresponding rules in the system \(NJ\).

\((LJ \rightarrow NJ_2)\) The structural rule cut is presented as a concatenation
operation, which is not an explicit rule in the system \(NJ\). (In the systems
of natural deduction there is an operation, concatenation operation
\((\Sigma/\Gamma/\Pi)\) (see [11, 12]), where \(\Sigma\) is a sequence of formula-trees and \(\Pi\)
is a formula-tree and \(\Gamma\) is a set of top formulae from \(\Pi\). The result of
\((\Sigma/\Gamma/\Pi)\) is a new derivation which is made by writing \(\Sigma\) above each top
formula in \(\Pi\) that belongs to \(\Gamma\) (see [11] p.26).)

\((LJ \rightarrow NJ_3)\) The image of a left rule from the system \(LJ\) is the
corresponding elimination rule together with a concatenation operation in
the system \(NJ\).

\((NJ \rightarrow LJ)\) Maps from \(\text{Der}(NJ)\) to \(\text{Der}(LJ)\) have the following characteristic.

\((NJ \rightarrow LJ_1)\) The image of an elimination rule from the system \(NJ\) is the
the corresponding left rule together with a cut rule in the system \(LJ\).

The solutions of the Problem 1. In the papers in which connections
between systems of sequents and natural deduction systems were studied there
are maps which connected derivations of these systems (see for example [8,
10, 14]). In these connections the problems \((LJ \rightarrow NJ_1)\) and \((LJ \rightarrow NJ_2)\)
were solved in the following way. Systems of sequents and natural deduction
systems, which were considered and whose connections were studied, are