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Khovanov-Rozansky Homology and Topological Strings*

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Abstract. We conjecture a relation between the $sl(N)$ knot homology, recently introduced by Khovanov and Rozansky, and the spectrum of BPS states captured by open topological strings. This conjecture leads to new regularities among the $sl(N)$ knot homology groups and suggests that they can be interpreted directly in topological string theory. We use this approach in various examples to predict the $sl(N)$ knot homology groups for all values of N. We verify that our predictions pass some non-trivial checks.

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1. Introduction and Summary

During the past 20 years, topological field theories have been the source of the vigorous interaction between theoretical physics and pure mathematics, increasingly fruitful for both fields. One of the famous examples of topological field theories is a Chern-Simons gauge theory [1,2]. Observables in this theory are naturally associated with knots and links. Specifically, given an oriented knot, K, and a representation of the gauge group, R, one can construct a Wilson loop operator, $W_R(K)$, whose expectation value turns out to be a polynomial invariant of K, such as the Jones polynomial and its generalizations [2].

Here, we will be mainly interested in the case where $R = \square$ is the fundamental representation of $sl(N)$. The corresponding quantum invariant

$$P_N(q) = \langle W_\square(K) \rangle \tag{1.1}$$

is a one-variable specialization of the HOMFLY polynomial [3]. It can be determined by the $sl(N)$ skein relation

*Dedicated to the memory of F.A. Berezin.

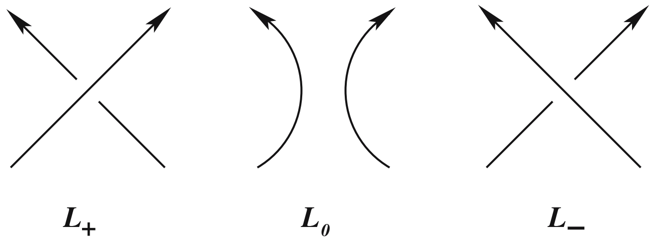

Figure 1. Link diagrams connected by the skein relation.

$$q^N P_N(L_+) - q^{-N} P_N(L_-) = (q^{-1} - q) P_N(L_0) \tag{1.2}$$

and by the normalization

$$P_N(\text{unknot}, q) = [N] = \frac{q^N - q^{-N}}{q - q^{-1}} \tag{1.3}$$

As shown in [4], Chern-Simons theory can be embedded in string theory. Later it was shown [5–7] using highly non-trivial 'stringy dualities' that this leads to a reformulation of quantum $sl(N)$ knot invariants in terms of topological string amplitudes (Gromov-Witten invariants). Moreover, it was realized that the polynomial invariants $P_N(q)$ can be reformulated in terms of integers $N_{\square,Q,s}$ which capture the spectrum of BPS states in the string Hilbert space [7,8]: [1]

$$P_N(q) = \frac{1}{q - q^{-1}} \sum_{s,Q \in \mathbb{Z}} N_{\square,Q,s} q^{NQ+s} \tag{1.4}$$

It is difficult to give a mathematically rigorous definition of BPS degeneracies. In the simpler case of closed topological strings, there is a similar notion of BPS degeneracies [6]. Attempts to give it a precise mathematical definition [9–13] led to a deeper understanding of Gromov-Witten invariants, though even there not all questions are answered. In the open string case, even though a mathematically rigorous proof of (1.4) is not available this equation is strongly supported by physical arguments and highly non-trivial checks, in particular, by the fact that considering it as a definition of $N_{\square,Q,s}$ one always obtains integers.

In another line of development, the quantum $sl(N)$ invariant $P_N(q)$ was lifted to a homological knot invariant [14–17]. For a given knot K and a fixed value of N, the $sl(N)$ homological invariant is a doubly graded cohomology theory, $\mathcal{H}_N^{i,j}(K)$, whose Euler characteristic with respect to one of the gradings equals the quantum $sl(N)$ invariant,

$$P_N(q) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}_N^{i,j}(K) \tag{1.5}$$

[1]In our normalization, the integer invariants $N_{\square,Q,s}$ are non-trivial only for even values of s.