Quantization: Deformation and/or Functor?

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Abstract. After a short presentation of the difference in motivation between the Berezin
and deformation quantization approaches, we start with a reminder of Berezin's view of
quantization as a functor followed by a brief overview of deformation quantization in
contrast with the latter. We end by a short survey of two main avatars of deformation
quantization, quantum groups and quantum spaces (especially noncommutative geometry)
presented in that perspective.

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1. Introduction

In the 1970s, the last decade of his too short life, the main subject of interest
for Alik Berezin was [63] to develop “supermathematics”, of which he was (mildly
speaking) one of the founders. However, at the same time and apparently as part
of a larger goal, he addressed the question of understanding quantization in as
precise a way as possible. The pursuit of precision was a constant feature in all
his works. It is essential in modern mathematical physics, but should be exercised
with care. He had tackled the issue of quantization before, especially in relation
with second quantization [5].

During the same time, independently, Moshé Flato and a number of coworkers
(among which I am happy to have been one) had exactly the same objectives,
with the opposite order of priority. Moshé and I had met Alik a couple of
times (in 1966 and 1972 in Moscow if my memory is correct) and appreciated him.
Those were difficult times as far as the communication between East and West was
concerned. As an unfortunate consequence, neither of us was really aware of the
efforts of the other group until about 1979–1980. We “flirted with” supersymmetry
in 1967–1970 [43, 44] before the terminology was coined, but that was from the
point of view of (traditional) Lie group theory, introducing a “Poincaré–Weyl” group with both vector and spinor translations. In the course of developing some possible applications we had, in 1970, to multiply the spinorial translations by an anticommuting operator $F$, in effect writing the Poincaré supersymmetry before the notion was introduced. But (except as a tool, e.g. with $osp(1, 2)$, for singletons) we did not pursue in that direction further: our main focus had shifted towards more elaborate topics in representation theory (integrability to Lie groups, nonlinear representations) and especially towards the deformation theory and what eventually was called deformation quantization.

There is, however, a deeper reason of scientific origin for the “orthogonality” of these efforts and different focuses of interest: Our approaches, in spite of an apparent similitude in techniques (in particular the use of phase space methods), have very different starting points.

Indeed, a scientist should answer three questions: why, what and how. It is generally recognized that work is 1% inspiration and 99% perspiration. The latter applies to how. Knowing what one is doing is very important and represents the greater part of the inspiration. But it is essential to know why one pursues such a research. That is where our two approaches differed in an essential way.

Mathematics and physics are two communities separated by a common language. The way of using mathematical formalism is generically very different in both. Not many can speak at ease that common language with both accents and styles. Alik was basically a mathematician, even if he was deeply interested in physics, motivated by it, knew a lot of physics and could even [63] speak with physicists using their language – albeit probably not with their accent. (This surfaced occasionally in some pedantism, see e.g. [9], when dealing with expressions commonly used “at the physical level of rigor” but rather heuristic as far as mathematical rigor is concerned.) In particular, like almost all mathematicians (and physicists for that matter) he took as a God-given axiom the operator formulation of quantum theories that is common fare among physicists. His aim was therefore to understand mathematically the correspondence principle. That led him, e.g. in his beautiful papers [6–8] to which we really paid attention a number of years later, to consider quantization (roughly speaking, his formulation is as usual more precise) as a functor between a category of algebras of classical observables (on phase space) and a category of algebras of operators (in Hilbert space). That is very nice, but it is a kind of translation of the physical reality (as expressed by most physicists) into mathematical language, while supermathematics was for him a much more challenging enterprise, especially as a mathematician.

Flato knew first-hand, having interacted in Israel with top level physicists and mathematicians already as an undergraduate student, that physicists are neither God nor Jesus, but that at least the best of them (e.g. Dirac with his “delta function”) sense very well where are the stones under the water surface when they walk over mathematical waters. Mathematicians would try to build a bridge following the physicists’ path (the theory of distributions in the above simple example), often