Letters in Mathematical Physics (2007) 81:1–17
DOI 10.1007/s11005-007-0166-y © Springer 2007

Coherent States for Hopf Algebras

ZORAN ŠKODA
Theoretical Physics Division, Institute Rudjer Bošković, P.O.Box 180, 10002 Zagreb, Croatia. e-mail: zskoda@irb.hr

Received: 7 July 2003 / Revised: 16 January 2007 / Accepted: 22 May 2007
Published online: 22 June 2007

Abstract. Families of Perelomov coherent states are defined axiomatically in the context of unitary representations of Hopf algebras. A global geometric picture involving locally trivial noncommutative fibre bundles is involved in the construction. If, in addition, the Hopf algebra has a left Haar integral, then a formula for noncommutative resolution of identity in terms of the family of coherent states holds. Examples come from quantum groups.

Mathematics Subject Classification (2000). 14A22, 16W30, 14L30, 58B32.

Keywords. coherent states, Hopf algebra, comodule algebra, Ore localization, localized coinvariants, line bundle, resolution of unity.

Perelomov's construction of coherent states [20,21] here extended to a class of Hopf algebras, classically goes as follows: given a real Lie group G, and a unitary irreducible representation $T : G \to \mathrm{Aut}\, V$ on a complex Hilbert space V, fix a vector v_0 in V with projective isotropy subgroup $H \subset G$ (i.e. $h \in H$ iff hv_0 equals v_0 up to a constant phase). There is a unitary character $\chi : H \to S^1$ such that $hv_0 = \chi(h)v_0$ for each $h \in H$. For G compact, the representation T extends to a representation of the complexification $G^{\mathbb{C}}$ of G.

A **family of Perelomov coherent vectors** in V is a family of vectors $\{C(u), u \in G/H\}$, such that $C([g]) = T(g)v_0$ up to a phase. Coherent *states* are projective classes (rays) of coherent *vectors*, but in practice one often says "coherent states" for both notions. If V is constructed by the method of geometric quantization, i.e. as the space of holomorphic sections ΓL of the corresponding quantization line bundle L over G/H, then the coherent vectors may be defined invariantly in terms of that line bundle [22]. For G a compact form of a semisimple Lie group $G^{\mathbb{C}}$, the details are in Sect. 3 below.

Hopf algebras appear in physics as symmetries of noncommutative and quantum spaces [6,15–18,37]. Algebra $\mathcal{O}(G) = \Gamma\mathcal{O}_G$ of regular functions on affine algebraic group G are commutative examples of Hopf algebras [11] with coproduct $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$ given by $(\Delta f)(g_1, g_2) = f(g_1 \cdot g_2)$. In fact, the category of commutative Hopf algebras is antiequivalent to the category of affine group schemes [11,26]. Hence, the noncommutative Hopf algebras are thought of

as (duals to) noncommutative affine group schemes ([8,26]; drawback: $\otimes$ is not a categorical product of noncommutative rings). Actions of affine group schemes generalize then to the coactions of Hopf algebras, which can furthermore be "structure groups" of noncommutative fibre bundles. The total space of such a bundle is either a single algebra (affine case) or a more complicated system of algebras or categories with gluing or localizing mechanism to pass between global and local description. Noncommutative fibre bundles with coacting Hopf algebras playing the role of a structure group first appeared in now classical work on smash products and Hopf–Galois extensions.

Then, H-J. Schneider introduced in [27] a crucial descent theorem supporting the geometric torsor intuition for faithfully flat Hopf–Galois extensions. In a study of noncommutative algebras equipped with differential calculi, Majid and Brzeziński [5] discovered a remarkable condition on differential calculi which enters the natural definition of principal bundles in that case. The coherent states on noncommutative projective homogeneous spaces, exhibited in the present work, seem to need a bundle theory extended in a different direction. To this aim, the present author has extended the concepts of Zariski *locally trivial* principal fibre bundles ([28,29, Škoda, Z. in Quantum bundles using coactions and localization, in preparation], and [33], Part I) to the setup where both total and base space are noncommutative (described locally by noncommutative algebras) and *not necessarily affine*.

Every complex semisimple Lie group $G^{\mathbb{C}}$ is an affine algebraic $\mathbb{C}$-group, and $G^{\mathbb{C}} \to G^{\mathbb{C}}/B$ is an algebraic principal fibration Zariski locally trivialized in a cover by shifts by action of Weyl group W of the main Bruhat cell [11]. Noncommutative analogues of such fibrations, derived from quantum matrix groups $\mathcal{G}_q$, are recently exhibited [28,29]. The fibrations trivialize in coaction-compatible Ore localizations $S_w^{-1}\mathcal{G}_q$ labeled by the elements w of the Weyl group W. The trivializations are explicitly computed using an elaborate *Ansatz* involving q-w-Gauss decompositions ([29], Theorems 9–12; proofs in [28] and [33], II).

In noncommutative case, it is not appropriate to seek for individual coherent vectors or rays in representation space V. A *family* of coherent vectors C should be a section of a noncommutative bundle $V \otimes L_\chi$ over a noncommutative "coset" space X "parametrizing would-be individual" coherent states, where the fiber $V = V_\chi = \Gamma L_\chi = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} \mathbb{C}_\chi$ is an analogue of a holomorphically induced representation space, χ is an analogue of a character of the inducing subgroup B and L_χ is an analogue of the Borel–Weil line bundle. Our noncommutative coset spaces are patched from charts. Local descriptions of X and C in different *covers* by charts are naturally equivalent. Earlier studies of coherent states for quantum groups [14, 25] used computations in a single local chart. One of our goals was to show that states locally computed in [25] may be defined *a priori*, regardless coordinate choices. The main goal was to find a resolution of unity in terms of coherent states of compact quantum groups.