Abstract Kelvin–Noether Theorems for Lie Group Extensions

CORNELIA VIZMAN
Department of Mathematics, West University of Timisoara, Bd. V.Parvan 4, 300223 Timisoara, Romania. e-mail:vizman@math.uvt.ro

Abstract. We present an abstract Kelvin–Noether theorem for geodesic equations on abelian Lie group extensions with right invariant metrics and we apply it to equations of hydrodynamical type. Another Kelvin–Noether theorem for a class of central extensions of semidirect products is shown.

Mathematics Subject Classification (2000). 58D05, 35Q35.

Keywords. Kelvin–Noether quantity, central and abelian extension, Euler–Poincaré equation, geodesic equation.

1. Introduction

Kelvin circulation theorem in ideal hydrodynamics, saying that the vorticity is transported by the flow, can be interpreted as a Noether theorem [1,9].

Let $G$ be a regular Fréchet Lie group [6] with Lie algebra $\mathfrak{g}$ and let $L: TG \to \mathbb{R}$ be a right invariant Lagrangian with value $\ell: \mathfrak{g} \to \mathbb{R}$ at the identity. By $\frac{\delta \ell}{\delta u}$ we denote the functional derivative of $\ell$ and by $\text{ad}^*$ the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$:

$$\left(\frac{\delta \ell}{\delta u}, v\right) = \lim_{\varepsilon \to 0} \left[ \ell(u + \varepsilon v) - \ell(u) \right], \quad (\text{ad}^*(u)m, v) = -(m, \text{ad}(u)v), \quad \forall v \in \mathfrak{g}.$$

Here the bracket $(,)$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

The curve $g$ in $G$ satisfies the Euler–Lagrange equation for the right invariant Lagrangian $L$ if and only if its right logarithmic derivative $u = g'g^{-1}$ satisfies the equation

$$\frac{d}{dt} \frac{\delta \ell}{\delta u} = \text{ad}^*(u) \frac{\delta \ell}{\delta u},$$

(1)
called the right Euler–Poincaré equation [8].

Considering a $G$-manifold $C$ and a $G$-equivariant map $\kappa : C \to \mathfrak{g}^*$, an abstract Noether theorem for the Euler–Poincaré equation (1) can be formulated.
THEOREM 1 [4]. The Kelvin quantity

\[ I : \mathcal{C} \times g \to \mathbb{R}, \quad I(c, u) = \left( \kappa(c), \frac{\delta \ell}{\delta u} \right) \]

is conserved along (1) in the sense that \( I(t) = I(g(t) \cdot c_0, u(t)) \) is time independent for any curve \( g \) in \( G \) whose right logarithmic derivative \( u = g'g^{-1} \) is a solution of (1) and for any \( c_0 \in \mathcal{C} \).

To recover Kelvin circulation theorem in ideal hydrodynamics one has to take \( G = \text{Diff}_{\text{vol}}(M) \), the Fréchet Lie group of volume preserving diffeomorphisms of a compact Riemannian manifold \((M, g)\), so \( g = \mathfrak{x}_{\text{vol}}(M) \) is the Lie algebra of divergence free vector fields with opposite bracket. Denoting by \( \mu \) the volume form on \( M \), the regular dual is \( g^* = \Omega^1(M)/dC^\infty(M) \) with pairing \( ([\alpha], u) = \int_M \alpha(u)\mu \) and coadjoint action is \( \text{ad}^*(u)[\alpha] = -[L_u\alpha] \).

The Euler–Poincaré equation corresponding to \( \ell(u) = \frac{1}{2} \int_M g(u, u)\mu \) is

\[ \partial_t [u^b] = -L_u[u^b], \quad \text{div} u = 0 \]

(2)
because \( \frac{\delta \ell}{\delta u} = [u^b] \). One can rewrite it as the Euler equation for ideal flow

\[ \partial_t u = -\nabla u - \text{grad} p, \quad \text{div} u = 0. \]

Considering the space \( \mathcal{C} \) of loops in \( M \) with natural action of \( G = \text{Diff}_{\text{vol}}(M) \), the map \( \kappa : \mathcal{C} \to g^{**} \) given by integration:

\[ (\kappa(c), [\alpha]) = \int_c \alpha, \quad c \in \mathcal{C}, \quad [\alpha] \in g^* \]

is well-defined and \( G \)-equivariant. The associated Kelvin quantity being

\[ I : \mathcal{C} \times g \to \mathbb{R}, \quad I(c, u) = \int_c u^b, \]

(4)
Theorem 1 assures that \( I(t) = \int g(t) \cdot c_0, u(t)^b \) is constant along ideal fluid flow. When the loop \( c_0 \) is the boundary of a surface \( \sigma_0 \), then the Kelvin quantity can be written in terms of the vorticity 2-form \( \omega = du^b \) as \( I(t) = \int g(t) \cdot \sigma_0 \omega(t) \). The time independence of the Kelvin quantity is equivalent to the time independence of \( g(t)^*\omega(t) \), i.e. the vorticity is transported by the flow.

The abstract Noether theorem is generalized in [4] to a Kelvin–Noether theorem for Euler–Poincaré equations with advected parameters. It is applied to finite dimensional mechanical systems, as well as to continua: heavy top, compressible magneto-hydrodynamics and Maxwell fluid. The Kelvin–Noether theorem can be adapted also to a more general situation: the affine Lagrangian semidirect product theory [3].

Geodesic equations on Lie groups with right invariant (weak) Riemannian metrics are obtained as right Euler–Poincaré equations for quadratic Lagrangians