Constant Mean Curvature Spacelike Surfaces in Three-Dimensional Generalized Robertson–Walker Spacetimes

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Abstract. Several uniqueness and non-existence results on complete constant mean curvature spacelike surfaces lying between two slices in certain three-dimensional generalized Robertson–Walker spacetimes are given. They are obtained from a local integral estimation of the squared length of the gradient of a distinguished smooth function on a constant mean curvature spacelike surface, under a suitable curvature condition on the ambient spacetime. As a consequence, all the entire bounded solutions to certain family of constant mean curvature spacelike surface differential equations are found.

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1. Introduction

Spacelike hypersurfaces of constant mean curvature (CMC) in an \( n \) \((\geq 3)\)-dimensional spacetime are critical points of the area functional under a certain volume constraint [8]. When the ambient spacetime is the Lorentz–Minkowski spacetime \( \mathbb{L}^n \), many results have been obtained from different viewpoints. For instance, Aiyama [1] and Xin [22] (see also [16] for a first weaker version given by Palmer) obtained independently a characterization of spacelike hyperplanes as the only complete CMC spacelike hypersurfaces in \( \mathbb{L}^n \) whose hyperbolic image is bounded,

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by applying the generalized maximum principle due to Omori-Yau \cite{15,21} and the Calabi–Bernstein theorem \cite{10}. As another application, Aledo and Alias \cite{3} characterized the spacelike hyperplanes as the only complete CMC spacelike hypersurfaces in $\mathbb{L}^n$ which lie between two parallel spacelike hyperplanes.

In this paper, we will deal with spacelike surfaces but in a more general setting. Indeed, we will consider CMC spacelike surfaces in three-dimensional generalized Robertson–Walker (GRW) spacetimes. Recall that an $n$-dimensional GRW spacetime is a warped product of a definite negative one-dimensional base, and a Riemannian manifold as a fiber \cite{5}. It is important to note that the family of GRW spacetimes includes classical Robertson–Walker (RW) spacetimes, i.e. the case in which the fiber has constant sectional curvature. Moreover, small deformations of the metric on the fiber of a RW spacetime also fit into the class of GRW spacetimes. Thus, GRW spacetimes are not necessarily spatially homogeneous, in opposition to the classical cosmological models. Spatial homogeneity seems appropriate just as a rough approach to consider the universe in the large. However, in order to consider it in a more accurate scale, this assumption could not be realistic to model universes with inhomogeneous spacelike geometry \cite{18}. On the other hand, observe that a conformal change of the metric of a GRW spacetime produces a new GRW spacetime if the conformal factor only depends on the time coordinate. Finally, let us remark that although three-dimensional spacetimes are too unrealistic, they have been deeply studied from a purely geometric viewpoint. In fact, they can be used to light suitable extensions of geometric properties to usual four-dimensional relativistic models.

Any GRW spacetime possesses a timelike, conformal and closed vector field $\xi$ (see Section 2), which is a useful tool for the study of spacelike hypersurfaces. Indeed, several uniqueness and non-existence results for compact CMC spacelike hypersurfaces in $n$-dimensional GRW spacetimes, and spacetimes with other infinitesimal symmetries, have been obtained in \cite{5–7}. The compactness of the hypersurface leads to the compactness of the fiber, i.e. the GRW spacetime must be spatially closed. Conversely, under certain natural assumptions, a complete spacelike hypersurface in a spatially closed GRW spacetime must be compact \cite{5}. When the fiber is not necessarily compact, uniqueness results for non-compact CMC spacelike hypersurfaces in GRW spacetimes have been given, under certain boundedness assumption on the unit normal vector field \cite{13}.

For the case of maximal surfaces, i.e. CMC spacelike surfaces with zero mean curvature, in certain three-dimensional RW spacetimes with fiber $\mathbb{R}^2$ and non-locally constant warping function, new Calabi–Bernstein problems for the maximal surface equation were stated \cite{12}. They were later widely generalized for the case of CMC spacelike surfaces in \cite{20}. Even more, if the fiber of the three-dimensional GRW spacetime $M$ is not assumed to be $\mathbb{R}^2$, but $M$ obeys certain natural relativistic energy condition, complete maximal surfaces have been classified, and other new Calabi–Bernstein problems for the maximal surface equation have been solved \cite{9}.