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L_∞-Algebras from Multisymplectic Geometry

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Abstract. A manifold is multisymplectic, or more specifically n-plectic, if it is equipped with a closed nondegenerate differential form of degree $n+1$. In previous work with Baez and Hoffnung, we described how the 'higher analogs' of the algebraic and geometric structures found in symplectic geometry should naturally arise in 2-plectic geometry. In particular, just as a symplectic manifold gives a Poisson algebra of functions, any 2-plectic manifold gives a Lie 2-algebra of 1-forms and functions. Lie n-algebras are examples of L_∞-algebras: graded vector spaces equipped with a collection of skew-symmetric multi-brackets that satisfy a generalized Jacobi identity. Here, we generalize our previous result. Given an n-plectic manifold, we explicitly construct a corresponding Lie n-algebra on a complex consisting of differential forms whose multi-brackets are specified by the n-plectic structure. We also show that any n-plectic manifold gives rise to another kind of algebraic structure known as a differential graded Leibniz algebra. We conclude by describing the similarities between these two structures within the context of an open problem in the theory of strongly homotopy algebras. We also mention a possible connection with the work of Barnich, Fulp, Lada, and Stasheff on the Gelfand–Dickey–Dorfman formalism.

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1. Introduction

Multisymplectic manifolds are smooth manifolds equipped with a closed, nondegenerate differential form. In this paper, we call such a manifold 'n-plectic' if the form has degree $n+1$. Hence, a 1-plectic manifold is a symplectic manifold. Multisymplectic geometry originated in covariant Hamiltonian formalisms for classical field theory, just as symplectic geometry originated in classical mechanics (See, for example, [11,18,19,22], as well as the review article [31].). More specifically, in $(n+1)$-dimensional classical field theory, one can construct a finite-dimensional $(n+1)$-plectic manifold known as a 'multi-phase space'. Particular submanifolds of this space correspond to solutions of the theory. The data encoded by the

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submanifolds include the value of the field as well as the value of its 'multi-momentum' at each point in space-time. The multi-momentum is a quantity that is related to the time and spatial derivatives of the field via a Legendre transform, in a manner similar to the relationship between the velocity of a point particle and its momentum. In fact, a $(0+1)$-dimensional theory is just the classical mechanics of point particles, and the corresponding 1-plectic manifold is the usual extended phase space whose points correspond to time, position, energy, and momentum.

However, multisymplectic manifolds can be found outside the context of classical field theory and are interesting from a purely geometric point of view. For motivation, we provide the following examples:

- An $(n+1)$-dimensional orientable manifold equipped with a volume form is an n-plectic manifold.
- Given a manifold M, the nth exterior power of the cotangent bundle $\Lambda^n T^*M$ admits a canonical closed non-degenerate $(n+1)$-form and therefore is an n-plectic manifold. This is a generalization of the canonical symplectic structure on the cotangent bundle.
- Any compact simple Lie group G is a 2-plectic manifold when equipped with the canonical bi-invariant 3-form

  $$\nu(x, y, z) = \langle x, [y, z] \rangle,$$

  where $x, y, z \in \mathfrak{g}$ and $\langle \cdot, \cdot \rangle$ is the Killing form. The relationship between this 2-plectic manifold and the topological group $\mathrm{String}(n)$, which arises in the study of spin structures on loop spaces, can be found in our previous work with Baez [5].
- Let (M, g) be a Riemannian manifold which admits two anti-commuting, almost complex structures $J_1, J_2 \colon TM \to TM$, i.e., $J_1^2 = J_2^2 = -\mathrm{id}$ and $J_1 J_2 = -J_2 J_1$. Then $J_3 = J_1 J_2$ is also an almost complex structure. If J_1, J_2, J_3 preserve the metric g, then one can define the 2-forms $\theta_1, \theta_2, \theta_3$, where $\theta_i(v_1, v_2) = g(v_1, J_i v_2)$. If each θ_i is closed, then M is called a hyper-Kähler manifold [36]. Given such a manifold, one can construct the 4-form:

  $$\omega = \theta_1 \wedge \theta_1 + \theta_2 \wedge \theta_2 + \theta_3 \wedge \theta_3.$$

  It is straightforward to show ω is closed and nondegenerate. Hence, a hyper-Kähler manifold is a 3-plectic manifold [10].

More examples, as well as the multisymplectic analogs of isotropic submanifolds, co-isotropic submanifolds and real polarizations can be found in the papers by Cantrijn et al. [10] and Ibort [20].

In our previous work with Baez and Hoffnung [4], we described how 2-plectic geometry can be understood as higher or 'categorified' symplectic geometry. For example, if a symplectic structure is integral, then it corresponds to the curvature of a principal $U(1)$-bundle. Similarly, in the 2-plectic case, the integrality condition