Abelian Quiver Invariants and Marginal Wall-Crossing

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Abstract. We prove the equivalence of (a slightly modified version of) the wall-crossing formula of Manschot, Pioline and Sen and the wall-crossing formula of Kontsevich and Soibelman. The former involves abelian analogues of the motivic Donaldson–Thomas type invariants of quivers with stability introduced by Kontsevich and Soibelman, for which we derive positivity and geometricity properties.


Keywords. quivers, abelian representations, motivic invariants, wall-crossing.

1. Introduction

In [12,13] a general framework for the study of the wall-crossing behaviour of Donaldson–Thomas invariants for 3-Calabi–Yau categories was described. It was observed in [13] that motivic Donaldson–Thomas invariants of 3-Calabi–Yau manifolds should be equivalent to the refined BPS invariants (see also [4]). Although the primary interest lies in Donaldson–Thomas invariants for 3-Calabi–Yau categories, it turns out that Donaldson–Thomas invariants for quivers play a key role in this setup, since they universally capture the behaviour of Donaldson–Thomas invariants when crossing a single wall. An alternative definition of motivic Donaldson–Thomas invariants of quivers, based on cohomological Hall algebras, was introduced in [14]. Their definition conforms to the proposed mathematical definition of the BPS state algebra of [9]. Central to the setup of [14] are various integrality and positivity predictions for the motivic invariants.

The relative integrality of Donaldson–Thomas invariants for quivers was established in [20], based on methods [18,19]. These methods were combined in [22] with the tropical vertex methods of [8] to relate topological invariants of quiver moduli to Gromov-Witten invariants of toric surfaces (see [6] for a refined/quantum analogue).

Recently Manschot et al. [16] proposed an explicit wall-crossing formula (called MPS wall-crossing in the following) for the BPS invariants in the rank two lattice
of charges by using multi-cantered black hole solutions in supergravity. They also
conjectured that their formula is equivalent to the Kontsevich–Soibelman (KS)
wall-crossing formula [13] in the refined case and to the Joyce–Song wall-crossing
formula [12] in the unrefined case. In this paper, we will show that a slightly mod-
ified MPS formula is equivalent to the KS wall-crossing formula.

A key ingredient of the MPS wall-crossing formula are abelian analogues of the
motivic Donaldson–Thomas invariants of [14]. In the language of [16], these arise
since the index of certain quivers can be reduced to the abelian case, by a phys-
ical argument which allows trading Bose–Fermi statistics with its classical limit,
Maxwell–Boltzmann statistics. The precise mechanism of this reduction, the MPS
degeneration formula, is again equivalent to the Harder–Narasimhan recursion of
duality of [22], the MPS degeneration formula is shown in [21] to correspond to
degeneration formulas in Gromov–Witten theory.

The importance of the approach [16] to the understanding of wall-crossing
formulas and motivic invariants for quivers makes it desirable to study abelian
quiver invariants more systematically, which is the motivation for the present
paper.

One of our aims is to prove integrality and positivity properties of these and
related invariants, and confirm a hypothesis of [17] on their geometric nature.

Let us describe first the KS wall-crossing formula (or HN recursion). Let \( \Gamma \) be
a rank 2 lattice with a non-degenerate, integer valued skew-symmetric form \( \langle \cdot, \cdot \rangle \)
and let \( \Gamma_+ \subset \Gamma \) be a monoid having two generators. We define a total preorder
on \( \Gamma_+ = \Gamma \setminus \{0\} \) by setting \( \alpha \leq \beta \) if \( \langle \alpha, \beta \rangle \geq 0 \). Similarly we order rays
\( l = \mathbb{R}_{>0} \gamma \subset \Gamma \otimes \mathbb{R} \) with \( \gamma \in \Gamma_+ \). Assume now that we have two families of (refined, rational DT)
invariants \( \bar{\omega}_{-}^{+} \gamma \), \( \bar{\omega}_{+}^{+} \gamma \) for \( \gamma \in \Gamma_+ \), which are related by an equation of ordered prod-
ucts over rays taken in clockwise (resp. anticlockwise) order

\[
\prod_{l} \exp \left( \sum_{\gamma \in \cap \Gamma} \bar{\omega}_{-}^{+} x^{\gamma} \right) = \prod_{l} \exp \left( \sum_{\gamma \in \cap \Gamma} \bar{\omega}_{+}^{+} x^{\gamma} \right)
\]  

(1)

in the quantum torus (see Section 5.1) of \( \Gamma \). This is the KS wall-crossing formula.
It allows us to recursively express the invariants \( \bar{\omega}_{+}^{+} \gamma \) in terms of the invariants \( \bar{\omega}_{-}^{-} \gamma \).

For any nonzero \( m : \Gamma_+ \to \mathbb{N} \) with finite support define \( \|m\| = \sum m(\alpha) \alpha \in \Gamma_+ \) and
\( m! = \prod m(\alpha)! \in \mathbb{N} \). Then we can write (cf. [16, Eq. 1.5])

\[
\bar{\omega}_{+}^{+} \gamma = \sum_{m: \Gamma_+ \to \mathbb{N} \atop \|m\| = \gamma} \frac{g(m)}{m!} \prod_{\alpha \in \Gamma_+} (\bar{\omega}_{-}^{-} \alpha)^{m(\alpha)}
\]  

(2)

for some invariants \( g(m) \). The computation of these invariants is recursive and is
rather difficult (see however [18]).

Manschot et al. [16] suggested the following description of the invariants \( g(m) \).
They first construct invariants \( g(\alpha_1, \ldots, \alpha_n) \) for non-parallel \( \alpha_i \in \Gamma \), then extend