On Divisorial Filtrations on Sheaves

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Abstract—In this paper, we generalize the notion of Poincaré series of a multi-index divisorial filtration corresponding to a collection of sigma-processes to the case of an arbitrary locally-free sheaf on the space of blow-ups of the complex plane $\mathbb{C}^2$. For an arbitrary sheaf, we establish a representation of the series in terms of topological invariants of the sheaf. In particular, for the sheaf of functions, this representation coincides with the Poincaré series obtained by Gusein–Zade and Delgado.

Key words: Poincaré series, sigma-process, divisorial filtration, exceptional divisor, sheaf of 1-forms, holomorphic form, blow-up, Laurent series, Chern class.

1. INTRODUCTION

In [1], Gusein–Zade and Delgado have calculated the Poincaré series of a multi-index filtration on the space of germs of functions depending on two variables; the filtration is defined by the orders of these functions on exceptional divisors of the space of multiple blow-ups of the origin. Besides functions, the pull-back map also lifts holomorphic 1-forms to the space of modifications. Hence a multi-index filtration on the space of germs of holomorphic 1-forms on $\mathbb{C}^2$ can be defined in a similar way. This filtration normally corresponds to the filtration on the space of global sections of the sheaf of 1-forms on the space of blow-ups.

Calculating the Poincaré series for this filtration seems to be a more difficult problem than that for functions. Hence, as an approximation, one can try to replace the space of global sections by the corresponding sheaf and to replace the dimension of $H^0$ by the Euler characteristic of the quotient sheaves, combining them into a generating function.

In particular, in this paper, we prove that, for the sheaf of functions, the series coincides with the one derived in [1]. Theorem 1 describes the generating function for an arbitrary locally-free sheaf on the space of blow-ups in terms of the characteristic classes of its restriction to exceptional lines. In particular, the solution of the problem for the sheaf of 1-forms is given by Theorem 2.

By $v$ we denote the element $(v_1, \ldots, v_s)$ of the lattice $\mathbb{Z}^s$. A partial ordering on $\mathbb{Z}^s$ is defined by the following rule: $v \leq w$ if each coordinate $v_i$ does not exceed the corresponding value of $w_i$. For such a pair $v, w \in \mathbb{Z}^s$, we define the upper bound, $\sup\{v, w\}$, as the minimal element of the lattice $\mathbb{Z}^s$ which is greater than or equal to $v$ and $w$.

Definition. A family of subspace $\{L(v) \mid v \in \mathbb{Z}^s\}$ such that

(1) if $v_1 \leq v_2$, then $L(v_1) \supset L(v_2)$;
(2) $L(v) \cap L(w) = L(\sup\{v, w\})$;
(3) $L(0) = L$, where $0 = (0, \ldots, 0)$,

is called a decreasing $s$-index filtration on the vector space $L$.

Suppose that $L(v)$ is an $s$-index filtration on the space $L$ and all quotient spaces $L(v)/L(v+1)$ (where $1 = (1, \ldots, 1)$) are finite-dimensional. Set $d(v) = \dim L(v)/L(v+1)$. Properties (2) and (3)
imply that the condition
\[ L(v_1, \ldots, v'_i, \ldots, v_s) = L(v_1, \ldots, v''_i, \ldots, v_s) \]
holds for all \( v'_i < v''_i \leq 0 \). Hence the filtration is defined by the collection of subspaces \( L(v) \) for elements \( v \) with nonnegative components.

Consider the space
\[ \mathcal{L} = \mathbb{Z}[t_1, \ldots, t_s, t_1^{-1}, \ldots, t_s^{-1}] \]
of formal Laurent series depending on \( s \) variables. Generally speaking, the elements of \( \mathcal{L} \) are expressions of the form \( \sum_{v \in \mathbb{Z}^s} k(v) \cdot t^v \), where the summation over \( \mathbb{Z}^s \) is unlimited. The space \( \mathcal{L} \) is not a ring but a module over the ring of polynomials. Suppose that
\[ Q(t_1, \ldots, t_s) = \sum_{v \in \mathbb{Z}^s} d(v) \cdot t^v. \]
Since
\[ d(v_1, \ldots, v'_i, \ldots, v_s) = d(v_1, \ldots, v''_i, \ldots, v_s) \]
for \( v'_i < v''_i \leq 0 \), it follows that
\[ P'(t_1, \ldots, t_s) = Q(t_1, \ldots, t_s) \cdot \prod_{i=1}^s (t_i - 1) \]
is a power series, i.e., it does not contain negative degrees of variables.

**Definition.** The series
\[ P_L(t_1, \ldots, t_s) = \frac{P'(t_1, \ldots, t_r)}{t_1 \cdots t_s - 1} \]
is called the Poincaré series of the multi-index filtration \( \{L(v)\} \) on the space \( L \).

This definition of the Poincaré series of multi-index filtration was introduced in [2].

Suppose that \( \pi: (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0) \) is an analytic map which is an isomorphism outside the origin and can be represented as the composition of \( s \) sequential sigma-processes, so that the exceptional divisor \( \mathcal{D} \) is the union of \( s \) irreducible components \( E_i \) and each of them is isomorphic to the complex projective line.

**Lemma 1.** The map \( \pi^* \) establishes an isomorphism between \( H^0(\mathbb{C}^2, \Omega^k_{\mathbb{C}^2}) \) and \( H^0(\mathcal{X}, \Omega^k_{\mathcal{X}}) \).

**Proof.** Suppose that \( w \) is a holomorphic \( k \)-form on \( \mathbb{C}^2 \) such that \( \pi^* w = 0 \). Then \( w = 0 \) outside the origin. Hence \( w = 0 \) and, therefore, \( \ker \pi^* = 0 \).

Suppose that \( \tilde{w} \) is a holomorphic \( k \)-form on the variety \( \mathcal{X} \). Since \( \pi^* \) is an isomorphism outside the exceptional divisor, then the holomorphic \( k \)-form \( w = (\pi^*)^{-1} \tilde{w} \) is well defined on \( \mathbb{C}^2 \setminus \{0\} \). The Hartogs theorem ensures that this form can be extended to a holomorphic \( k \)-form \( w \) on \( \mathbb{C}^2 \). Then \( \pi^* w \) and \( \tilde{w} \) coincide outside \( \mathcal{D} \) and, therefore, they coincide on the entire variety \( \mathcal{X} \). Thus, \( \text{Im} \pi^* = H^0(\mathcal{X}, \Omega^k_{\mathcal{X}}) \) and \( \pi^* \) is an isomorphism. \( \square \)

Let \( \mathcal{J}_i \) be a sheaf of ideals defining \( E_i \) in \( \mathcal{X} \). Set
\[ d_i^j = -(E_i \circ E_j), \quad M = (m_i^j) = D^{-1}, \]
where “\( \circ \)” stands for the intersection index. Note that, for \( i \neq j \), \( d_i^j \) can take only two values, 0 or 1. Further, let \( \overline{E_i} \) be the smooth part of the line \( E_i \), i.e., \( \overline{E_i} \) is the component \( E_i \) minus all intersection points with other components of the exceptional divisor.