Call Option Prices Based on Bessel Processes

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Received: 1 September 2008 / Accepted: 27 July 2009 / Published online: 7 August 2009
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Abstract As a complement to some recent work by Pal and Protter (2007, 2009), we show that the call option prices associated with the Bessel strict local martingales are integrable over time, and we discuss the probability densities obtained thus.

Keywords Bessel processes · Last passage times · Strict local martingale

AMS 2000 Subject Classifications 60J60

1 Introduction: Some General Remarks

1.1

Let \((M_t, t \geq 0)\) denote a continuous local martingale, taking values in \(\mathbb{R}_+\). To any \(K > 0\), we associate the process \((K - M_t)^+, \ t \geq 0\). It is not difficult to show, after localizing \(M\), that this process \((K - M_t)^+\) is a (bounded) submartingale, and, as a consequence, the function:

\[
m_K^+(t) = E\left[(K - M_t)^+\right], \ t \geq 0
\]
is increasing, and bounded (by $K$). The study of such functions, considered (essentially) as distribution functions, has been the subject of the Bachelier Course (Bentata and Yor 2008a, b), given by the second author. In particular, if $M_t \to t \to \infty 0$, there is the formula

$$E\left[F_t (K - M_t)^+\right] = KE\left[F_t 1_{\mathcal{G}_K \leq 0}\right] \quad (1)$$

which is valid for every $F_t \geq 0$, $(F_t)$ measurable, and $\mathcal{G}_K = \sup \{t : M_t = K\}$. See also Madan et al. (2008a, b, c) and Profeta et al. (2009).

1.2

The present paper is devoted to the study of the functions:

$$m^{(-)}_K(t) = E\left[(M_t - K)^+\right] = E\left[(K - M_t)^-\right], \quad t \geq 0,$$

which play an important role in option pricing, as $(m^{(-)}(t))$ is the European call price with strike $K$, and maturity $t$, associated with the local martingale $(M_t)$. If $(M_t)$ is a “true” martingale, then $(M_t - K)^+$ is a submartingale, hence $(m^{(-)}(t), t \geq 0)$ is increasing. On the other hand, if $(M_t, t \geq 0)$ is a strict local martingale, that is: a local martingale, which is not a martingale, then the function $(m^{(-)}_K)$ is not in general increasing, or even monotone.

1.3

The most well-known example of a strict local martingale is $M_t = 1/R_t$, where $(R_t, t \geq 0)$ denotes the $BES(3)$ process, starting from 1, or, by scaling, equivalently from any $r > 0$. Then, the study of $(m^{(-)}_K(t))$ in this particular case has been undertaken in a remarkable paper by Pal and Protter (2007, 2009); the results of which have strongly motivated the present paper.

In the present paper, we take up again the study of this function $(m^{(-)}_K(t))$ in this particular case; we show that:

$$\int_0^\infty dt \, m^{(-)}_K(t) < \infty.$$

Hence, up to a multiplicative constant $(m^{(-)}_K(t), t \geq 0)$ is a probability density on $\mathbb{R}_+$; we identify the Laplace transform of this probability, and describe it as the law of a certain random variable defined uniquely in terms of $BES(3)$ process. This is done thanks to the Doob $h$-transform understanding of $BES(3)$ (from Brownian motion, killed when hitting 0), combined with general identity (1). We refer the reader to Section 2 for precise statements. In Section 3, we develop the same kind of study but this time with $M_t = 1/R_{t}^{(\delta-2)}$, $t \geq 0$, where $(R_t, t \geq 0)$ denotes the $BES(\delta)$ process, with $\delta > 2$, starting from 1. In Section 4, we present the graphs of the corresponding functions $(m^{(-)}_K(t), t \geq 0)$.

1.4

To summarize, the main point of this work is to use the interpretation of the generalized Black-Scholes quantities in terms of last passage times (formula (1)) in...