Lyapunov type stability and Lyapunov exponent for exemplary multiplicative dynamical systems

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Abstract This paper presents analysis of Lyapunov type stability for multiplicative dynamical systems. It has been formally defined and numerical simulations were performed to explore nonlinear dynamics. Chaotic behavior manifested for exemplary multiplicative dynamical systems has been confirmed by calculated Lyapunov exponent values.

Keywords Multiplicative calculus · Lyapunov stability · Lyapunov exponent

1 Introduction

Chaotic behavior can be observed in systems behavior from all fields of science. Our interests concern chaos occurring in process of defects growth in materials. Models of fractal defects evolution presented in [1–3] link the energy uniformly distributed over fractal and its measure \( \nu_D \) using essential material characteristics, which is energy density \( a(D) \) depending on fractal dimension:

\[
\mathcal{E} = a(D)\nu_D.
\]

In order to verify this theoretical model, stochastic numerical simulations of breaking fibers in composite were performed and described in [2]. The idea we would like to examine is if growing defects may behave in a chaotic way. Describing the evolution of defects treated as fractals implies usage of multiplicative derivatives, because ordinary additive derivative of function depending on fractal dimension or measure does not exist. Therefore, multiplicative calculus presented in [4] and restored in [1, 3] must be applied.

The goal of the paper is chaos examination in multiplicative dynamical systems described with multiplicative derivatives. Derived and tested methods will be employed to systems of fractal defects evolution. In this paper, calculations are performed for multiplicative counterparts of well-known dynamical systems: Lorenz system, which we will name multiplicative Lorenz system. It can be described with multiplicative derivatives, and also with additive derivatives, using the relation between additive and multiplicative derivatives presented in [1]. Analysis of nonautonomous multiplicative Lorenz system described with additive derivatives has been executed using the Lyapunov stability theory [5] and presented in [6].

This paper contains derivation of stability theory of the Lyapunov type for system of autonomous multiplicative differential equations. Obtained formula is tested for multiplicative Lorenz system described with multiplicative derivatives.

This paper also contains a proposed definition of a Lyapunov exponent for the multiplicative dynam-
2 Stability of Lyapunov type

On the basis of the Lyapunov theorem about stability for ordinary differential equations [5], its counterpart for multiplicative dynamical system has been derived.

For the system of autonomous multiplicative differential equations:

\[
\frac{\pi x_j}{\pi t} = f_j(x_1, \ldots, x_n), \quad j = 1, \ldots, n
\]  
(1)
equilibria are calculated from:

\[
f_j(x_1, \ldots, x_n) = 1.
\]  
(2)
The solution close to fixed point \(x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})\) is described with multiplicative Lorenz system described with multiplicative derivative, which is determined by initial conditions. Accordingly, the left-hand side of (4) is always diagonal. Choosing the particular basis in which the matrix \(\left[\frac{\pi f_j(x_0)}{\pi x_k}\right]\) is also diagonal, we obtain:

\[
\ln \frac{\pi e_j(t)}{\pi x} = \lambda_j \rightarrow \frac{\pi e_j(t)}{\pi t} = e^{\lambda_j},
\]  
(8)
where \(\lambda_j\) denotes the corresponding \(j\)th eigenvalue. Application of the relationship between additive and multiplicative derivative [3]:

\[
\frac{\pi f(x)}{\pi x} = \exp \left\{ \frac{\pi f'(x)}{f(x)} \right\}
\]  
(9)
to (8) gives us an additive ordinary differential equation:

\[
\frac{\pi e_j(t)}{\pi t} = \exp \left\{ \frac{\pi \hat{e}_j(t)}{e_j(t)} \right\} = e^{\lambda_j} \implies \hat{e}_j(t) = \frac{1}{t} \lambda_j e_j(t),
\]  
(10)
where \(\hat{e}_j(t)\) is an ordinary additive derivative. Its solution equals:

\[
e_j(t) = e_j^{(0)} e^{\lambda_j t},
\]  
(11)
and logarithm of both sides of (6) equals:

\[
\epsilon_j(t) \ln \frac{\pi e_j(t)}{\pi t} = \sum_{k=1}^{n} \epsilon_k(t) \ln \frac{\pi f_j(x_0)}{\pi x_k}.
\]  
(7)

For every basis in (4), we calculate derivatives along the direction of coordinates. Therefore, the left-hand side of (7) is positive, value is unstable.

\[
\epsilon_j(t) \ln \frac{\pi e_j(t)}{\pi t} = \sum_{k=1}^{n} \epsilon_k(t) \ln \frac{\pi f_j(x_0)}{\pi x_k}.
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