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Synchronization and coupling of Mandelbrot sets

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Abstract The definitions of synchronization and coupling of two different Mandelbrot sets are introduced. By the nonlinear coupling method, the synchronization and coupling of two different Mandelbrot sets are achieved, which make one Mandelbrot set change to be another and also make two different Mandelbrot sets change to be the same one.

Keywords Synchronization · Coupling · Mandelbrot set

1 Introduction

At the beginning of the twentieth century, the complex dynamical systems were studied by Fatou [1] and Julia [2]. They considered the general complex polynomial iterated mappings, $f^n(z) = f^{n-1}(f(z))$, and took the limit $n \to \infty$. Following the footsteps of Fatou and Julia, Mandelbrot [3] found that the value of the parameter c in the map $z \to f(z) = z^2 + c$ determines the fate of the infinite iteration of the critical point of the map. He was the first to plot the set of all values of c that the orbit at $z = 0$ is bounded, which is now called Mandelbrot set; see Fig. 1.

At present, fractal has been extensively applied to nature and other fields [4–7] since it was introduced by Mandelbrot. Using its special properties, we can have a better comprehension on many complicated problems and phenomena about biology, physics and so on. Mandelbrot set is a classical fractal set and is studied deeply by people. For example, Christian Beck [8] showed that the generalized Mandelbrot set M of the complex map $z_{n+1} = z_n^r + c$ has a concrete physical meaning in terms of the classical particle movement. For all kick strengths $c \in M$, the velocity of the particle remains bounded, whereas for $c \notin M$ the velocity asymptotically diverges—the particle "falls of" the potential.

Recently, the authors have discussed the synchronization of Julia sets [9]. However, the study about Mandelbrot set is limited to its properties, drawing. According to practical requirement, we need to consider to synchronize or couple the systems behaviors described by the Mandelbrot set. Hence, it is necessary to study the synchronization and coupling of different Mandelbrot sets. In this paper, we give the definitions of synchronization and coupling of two different Mandelbrot sets and consider the changing and coupling between two different Mandelbrot sets. Then, by the nonlinear coupling method, the synchronization and coupling of two different Mandelbrot sets are achieved, which make one Mandelbrot set change to

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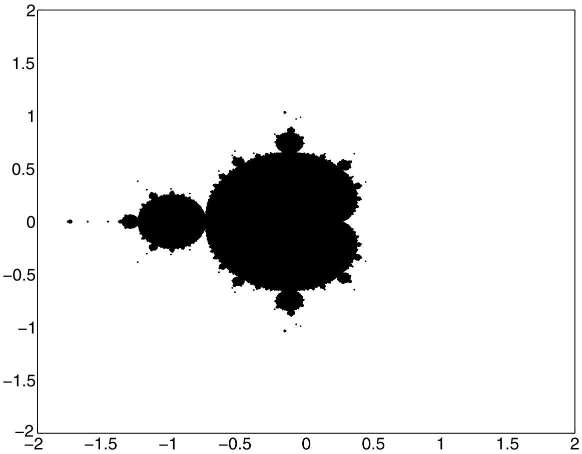

Fig. 1 Mandelbrot set M^c of $f(z_n;c)=z_n^2+c$

be another one and also make two different Mandelbrot sets change to be the same one.

2 Synchronization and coupling of Mandelbrot sets

Consider two systems in the complex domain $\mathbb{C}$

$$z_{n+1}=f(z_n;c), \tag{1}$$

and

$$w_{n+1}=g(w_n;d). \tag{2}$$

Obviously, the Mandelbrot sets of (1) and (2) are determinate and have not any relations between them. Denote them by M^c and M^d, respectively.

In order to associate M^c with M^d, we couple the systems (1) and (2). Introduce the coupled items into (1) and (2), then

$$z_{n+1}=f(z_n;c)+p(z_n,w_n;k), \tag{3}$$

$$w_{n+1}=g(w_n;d)+q(z_n,w_n;k), \tag{4}$$

where p and q are the coupling items and are the continuous maps about z_n and w_n, and k is the coupling strength. It is obvious that one Mandelbrot set is corresponding with a value of k and denote by M_k^c and M_k^d the Mandelbrot sets of systems (3) and (4), respectively. Coupling of Mandelbrot sets of (1) and (2) is taken to occur if

$$\lim_{k\to k_0}\left(M_k^c\cup M_k^d-M_k^c\cap M_k^d\right)=\emptyset \tag{5}$$

for some k_0 (k_0 may be ∞). Specifically, when $q(z_n,w_n;k)=0$, (5) becomes

$$\lim_{k\to k_0}\left(M_k^c\cup M^d-M_k^c\cap M^d\right)=\emptyset. \tag{6}$$

Then the synchronization of Mandelbrot sets of (1) and (2) is achieved. That is, the Mandelbrot set of (3) will change to be the Mandelbrot set of (2) when $k\to k_0$.

From (5), it is not difficult to see that the Mandelbrot sets of (3) and (4) will change to be the same one when $k\to k_0$ and from (6), the Mandelbrot set M_k^c will change to be the Mandelbrot set M^d when $k\to k_0$.

Now, we will introduce a nonlinear coupling method to achieve the synchronization and coupling of two different Mandelbrot sets.

We consider only the iterations which are bounded since $c\notin M$ if for some n_0, $f^{n_0}(0;c)$ is not bounded from the definition of Mandelbrot set. That is to say that there exists a positive real number T, we consider only the iterations included in the set $\{z:\ |z|\le T\}$. If the iterations of the coupled systems (3) and (4) are asymptotic, the boundedness of the iterations will be same and then the Mandelbrot sets of (3) and (4) will be same.

Take $p(z_n,w_n;k)=k[g(w_n;d)-f(z_n;c)]$ and $q(z_n,w_n;k)=0$, then system (3) becomes

$$z_{n+1}=f(z_n;c)+k\big[g(w_n;d)-f(z_n;c)\big], \tag{7}$$

which with (2) derives

$$z_{n+1}-w_{n+1}=(1-k)\big[f(z_n;c)-g(w_n;d)\big].$$

Therefore,

$$\begin{aligned}|z_{n+1}-w_{n+1}|&=|1-k|\big|f(z_n;c)-g(w_n;d)\big|\\&\le|1-k|(2T).\end{aligned} \tag{8}$$

If $|1-k|\to 0$ in the right side of (8), $|z_{n+1}-w_{n+1}|\to 0$. From the previous discussion, we know that the synchronization of the iterations is achieved and so the synchronization of Mandelbrot sets is achieved.

Take $f(z_n;c)=z_n^2+c$, $g(w_n;d)=w_n^4+d$, then Figs. 1 and 2 are their Mandelbrot sets, respectively.

It is easy to see from Fig. 3 that the Mandelbrot set M^c is changing to M^d with k increasing. When $k\to 1$ is big enough, it will turn to be M^d which illustrates that the synchronization of Mandelbrot sets M^c and M^d is achieved.