Nonlinear resonances in infinitely long 1D continua on a tensionless substrate

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Abstract In this work, we investigate the primary nonlinear resonance response of a one-dimensional continuous system, which can be regarded as a model for semi-infinite cables resting on an elastic substrate reacting in compression only, and subjected to a constant distributed load and to a small harmonic displacement applied to the finite boundary. By introducing a straightforward small amplitude expansion characterized by a smallness parameter $\varepsilon$ and by performing a Fourier analysis, we first determine the frequencies of the oscillations of the system about the static solution at all orders. We find that, at each order, there exists a critical (cutoff) frequency, above which the solution behaves as a traveling wave toward infinity, while it decays exponentially below it. We then examine the resonance response of the system when an external harmonic excitation is applied at the finite boundary. To this aim, we scale the external excitation with the third power of $\varepsilon$ and perform a Multiple-Time-Scale analysis, whose third-order consistency conditions give the differential equations which govern the behavior of the amplitude on the long time scale. In this way, we determine the third-order bending of the resonance curves, whose hardening or softening behavior depends upon the frequency of the chosen primary resonance.

Keywords Nonlinear oscillations · Wave equation · Klein–Gordon equation · Moving boundary problems · Multiple time scales expansions · Backbones · Hardening and softening behavior

1 Introduction

The nonlinear oscillations of continuous systems resting on a unilateral elastic substrate is a problem that arises in many practical applications and in various fields of engineering such as, for example, marine engineering (laying of off-shore pipelines [1, 2], mooring cables, etc.), railway engineering (soil-track and pantograph-power line dynamical interactions [3], etc.), mechanical engineering (laying of inshore pipelines, etc.), civil engineering (suspended bridges with no-pretensioned suspension, etc.).

Besides its practical relevance, the problem deserves also a theoretical interest, even in the simplest version where there are no geometric nonlinearities and where the supporting springs are piecewise linear. In fact, it involves non-smooth nonlinear dynamics.
and wave propagation [4, 5], it is an archetypal problem for soil-structure interaction [6], and mainly it is a moving boundary problem [7], whose interest and difficulty rest on the fact that the contact and non-contact domains are not known in advance, but they depend upon the solution itself.

In the past, several authors have investigated various aspects of the problem, both in a general framework [8–14] or by considering specific details, such as finite [14–16] or infinite (or semi-infinite) domains [17–19], harmonic in time and fixed in space [19, 20] or fixed in time and moving in space [21–23] loads, cables [17, 18, 23] or beams and plates [14,16, 19, 22, 24–26] mechanical models, numerical [19, 26, 27] or analytical approximate [16–18] solutions. We refer to the Introduction in [18] for a detailed description of the literature.

In this work, we continue our investigations on the nonlinear dynamics of the wave equation on a unilateral linearly elastic substrate [17–19, 27], which describes the nonlinear oscillations of straight cables (which is our reference mechanical problem) as well as the axial and torsional oscillations of rods. We consider a semi-infinite domain, while the actions on the cable are given by a static transversal uniformly distributed force, and a static displacement applied at the finite boundary, where a harmonic forcing excitation is also added. The only source of nonlinearity is given by the unilateral behavior of the springs, which can be thought as supporting only compression (e.g., in pipelines laying) or only tension (e.g., in suspended bridges). The other relevant mechanical feature is the unboundedness of the domains, which in certain cases allows for traveling waves which dissipate energy by radiation at infinity.

In [17, 18], the problem was studied by straightforward perturbation techniques, with the zero-order terms corresponding to the static solution. In [17], the first-order solution was obtained. Two different regimes were identified, one below (called subcritical) and one above (called supercritical) a certain critical (cutoff) excitation frequency. In the latter, energy is lost by radiation, while in the former this phenomenon does not occur and various resonances are observed instead; their number depends on the static configuration around which the system performs nonlinear oscillations. In the case of a beam, we have a qualitatively similar behavior, which has been numerically confirmed in [19]. In [18], the second-order behavior was analyzed in detail and the behavior of the higher-order terms was deduced as well. It was shown how the higher-order cutoff frequencies are related to the first-order one, a result which is not detectable by the first-order analysis. The secondary resonances were obtained, and their relationships with the primary resonances was discussed.

More complicated dynamical behaviors, including regimes with multiple Touch Down Points (TDPs—the points which divide the contact from the non-contact domains) have been investigated numerically in [27], showing the richness of the dynamical behavior of the considered system in spite of the various mechanical approximations mentioned above.

In this paper, the previous work of the authors is continued by considering for the first time the behavior of the system near primary resonances and by studying the bending of the resonant curves (backbones). This requires (i) the use of the multiple scale (MTS) method, and (ii) the analysis of the third order terms, both of which were not considered in the previous papers.

In fact, (i) the perturbation techniques used in [17] and [18] are not suited for analyzing typical nonlinear effects such as backbones; a different scaling must be used for the external excitation and the MTS method, a technique that has been extensively used to study the nonlinear oscillations of cables with different types of nonlinearities (see, e.g., [28]), is needed [29, 30]; (ii) the zero-order terms of our expansion correspond to the static solution [17, 18]; to first and second order, then we recover, as a free vibration problem, the results obtained in [17] and [18]. The third-order terms, which were not addressed before, contain the information on the bending of the resonant curves (backbones) and their study constitutes the main goal of this work.

An important (and partially unexpected) result of this analysis is that the system exhibits hardening (the backbone bends toward higher excitation frequencies) or softening (the backbone bends toward smaller excitation frequencies) behavior of the backbones depending on the value of the static displacement at the finite boundary, which is the only parameter governing the static configuration, and on the excitation frequency. A similar behavior has been observed, e.g., in [31], where the nonlinearity is of the hardening type for the first mode, whereas it is of the softening type for the second and higher modes. More precisely, our perturbative solution indicates that, if the frequency of