Estimates of the $l_2$ norm of the error in the conjugate gradient algorithm

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In this paper we derive a formula relating the norm of the $l_2$ error to the $A$-norm of the error in the conjugate gradient algorithm. Approximating the different terms in this formula, we obtain an estimate of the $l_2$ norm during the conjugate gradient iterations. Numerical experiments are given for several matrices.

**Keywords:** conjugate gradient, norm of the error

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1. Introduction

In this paper we derive a formula relating the $l_2$ norm of the error to the $A$-norm of the error in the conjugate gradient algorithm for solving linear systems with a symmetric positive definite matrix. The problem of computing estimates for the $A$-norm of the error was considered in [5–9]. This is summarized in [10]. The computation of estimates in finite precision arithmetic was studied in [11].

Let $A$ be a large and sparse symmetric positive definite matrix of order $n$ and suppose we have an approximate solution $\tilde{x}$ of the linear system

$$Ax = g,$$

where $g$ is a given right-hand side vector. The residual $r$ is defined as $r = g - A\tilde{x}$. The error $e$ being $e = x - \tilde{x}$, we obviously have, $e = A^{-1}r$. Therefore, if we consider the $A$-norm of the error,

$$\|e\|_A^2 = (e, Ae) = e^T A e = r^T A^{-1} A A^{-1} r = r^T A^{-1} r.$$

Here we are interested to use the $l_2$-norm, for which

$$\|e\|^2 = r^T A^{-2} r.$$
To solve the linear system we use the conjugate gradient (CG) algorithm: let \( x^0 \) be given, \( r^0 = g - Ax^0 \), \( p^0 = r^0 \), for \( k = 1, \ldots \) until convergence

\[
\begin{align*}
\gamma_k &= \frac{r^{k-1}^T r^{k-1}}{p^{k-1}^T A p^{k-1}}, \\
x^k &= x^{k-1} + \gamma_{k-1} p^{k-1}, \\
r^k &= r^{k-1} - \gamma_{k-1} A p^{k-1}, \\
\beta_k &= \frac{r^{k-1}^T r^{k-1}}{r^{k-1}^T p^{k-1}}, \\
p^k &= r^k + \beta_k p^{k-1}.
\end{align*}
\]

We would like to cheaply estimate the \( l_2 \) norm of the error, eventually some iterations before the current one.

The contents of the paper are as follows. In section 2 we derive a formula relating the \( l_2 \) norm to the \( A \)-norm of the error. Section 3 shows how to use this formula to compute estimates of the \( l_2 \) norm by introducing a delay. Section 4 gives some numerical experiments. In section 5 we comment on what can be done when introducing a preconditioner to speed up convergence. The last section gives some conclusions.

2. A formula for the norm of the error

Formulas were given in [5–9] to compute bounds or estimates for the \( A \)-norm of the error for the conjugate gradient (CG) method. It is well known that CG is closely related to the Lanczos algorithm. These computations used the formula

\[
(A \varepsilon^k, \varepsilon^k) = (r^0, A^{-1} r^0) - \| r^0 \| (T_k^{-1} e^1, e^1),
\]

where \( T_k \) is the matrix of the Lanczos algorithm coefficients and \( e^j \) is the \( j \)th column of the identity matrix. The relation for the matrix \( V_k \) of the Lanczos vectors is the following:

\[
AV_k = V_k T_k + \eta_k+1 \tilde{v}^{k+1} (e^k)^T.
\]

\( T_k \) is a tridiagonal matrix denoted as

\[
T_k = \begin{pmatrix}
\alpha_1 & \eta_2 & \eta_3 & \cdots & \eta_{k-1} & \alpha_{k-1} & \eta_k \\
\eta_2 & \alpha_2 & \eta_3 & \cdots & \eta_{k-1} & \alpha_{k-1} & \eta_k \\
\alpha_3 & \eta_3 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\eta_{k-1} & \alpha_{k-1} & \eta_k \\
\eta_k & \alpha_k
\end{pmatrix}.
\]

This can also be written as

\[
AV_k = V_{k+1} \tilde{T}_k,
\]