A Characterisation of Posets that Are nearly Antichains

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Abstract. We present a characterisation of posets of size \( n \) with linear discrepancy \( n - 2 \). These are the posets that are “furthest” from a linear order without being an antichain.

Mathematics Subject Classifications (2000):

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In [2], Fishburn, Tanenbaum and Trenk introduced the notion of linear discrepancy of a poset as a measure of its non-linearity. Let \( P = (X, \prec) \) be a finite poset. A pair of elements \( x, y \in X \) is said to be comparable, written \( x \perp y \), if either \( x \prec y \) or \( y \prec x \), and incomparable, written \( x \parallel y \), otherwise. A poset where every pair of elements is comparable is a chain, while a poset where every pair of elements is incomparable is an antichain. A disjoint union of posets \( P_1 = (X_1, \prec_1) \), \( P_2 = (X_2, \prec_2) \) is the poset \( P_1 + P_2 = (X_1 \cup X_2, \prec_1 \cup \prec_2) \), assuming without loss of generality that \( X_1, X_2 \) are disjoint sets. The notation \( m \) is used to denote the chain of \( m \) elements.

For any non-negative integer \( k \), a \( k \)-linear labelling of \( P \) is an injective function \( f : X \rightarrow \{1, \ldots, |X|\} \) such that \( f(a) < f(b) \) if \( a \prec b \), and \( |f(a) - f(b)| \leq k \) if \( a \parallel b \). The linear discrepancy \( ld(P) \) of \( P \) is the least \( k \) for which there exists a \( k \)-linear labelling of \( P \). Note that if \( P \) is a chain then \( ld(P) \) is 0. The results in [2] deal with the linear discrepancies of special classes of posets, including the standard examples, disjoint unions of chains, semiorders and Boolean lattices. In [4] the linear discrepancy of products of chains are determined. In [3], it is shown that the linear discrepancy of any poset \( P = (X, \prec) \) is equal to the bandwidth of the incomparability graph \( G_{\parallel}(P) = (X, E) \) of \( P \), where the edge set \( E \) consists of pairs of elements that are incomparable. Using this result, Rautenbach [6] provides solutions to some open problems formulated in [2], and poses a conjecture about a structural characterisation of the posets with linear discrepancy at most 2.

It is clear from the definition that \( 0 \leq ld(P) \leq n - 1 \), and \( ld(P) = 0 \) if and only if \( P \) is a chain, while \( ld(P) = n - 1 \) if and only if \( P \) is an antichain. In addition, the posets with \( ld(P) = 1 \) are characterised in [2] as precisely the semiorders of
width 2, that is, posets with none of \(3 + 1, 2 + 2\) or \(1 + 1 + 1\) as an induced subposet. In view of the use of linear discrepancy as a measure of non-linearity, these are the posets that are “nearly” chains. Here we present a characterisation of posets that are “nearly” antichains.

**Theorem 1.** Let \(P = (X, \prec)\) be a poset with \(|X| = n \geq 3\). Then \(\text{ld}(P) = n - 2\) if and only if \(P\) is a disjoint union of one or more of \(P_1, P_2, P_3, P_4\), but \(P \neq P_2\) or \(P_3\) or \(1 + \cdots + 1\), where

\[
P_1 = U \cup V, \quad \text{where} \quad U = \{u\}, \quad V = \{v_1, \ldots, v_h\}, \quad h \leq n - 1, \quad u \perp v_1 \text{ and } x \perp y \text{ only if } x = u, \ y \in V;
\]

\[
P_2 = 2;
\]

\[
P_3 = 3;
\]

\[
P_4 = 1.
\]

The poset \(P_1\) consists of the antichain \(V = \{v_1, \ldots, v_h\}\) together with the element \(u\) that is less than (or dually, more than) at least one of the elements in \(V\), and can be depicted as follows:

![Diagram of poset P1](image)

We prove a few intermediate results before proving Theorem 1. We use the notation \(X(i, j)\) (from [5]) to denote the elements of \(X\) with elevation \(i\) and depth \(j\), that is, the number of elements in a chain of maximum length lying under \(x \in X(i, j)\) is \(i\) and the number of elements in a chain of maximum length lying over \(x\) is \(j\).

**Lemma 2.** Let \(P = (X, \prec)\) be a poset with \(|X| = n \geq 3\) and \(\text{ld}(P) = n - 2\). If the longest chains in \(P\) have size \(h \geq 3\) then they are disjoint.

**Proof.** Suppose that \(k \geq 2\) longest chains \(C_1, \ldots, C_k\) of size \(h\) intersect. Consider the maximal elements

\[
X_{\text{max}} = X(h - 1, 0) \cap \{C_1 \cup \cdots \cup C_k\}
\]

and the minimal elements

\[
X_{\text{min}} = X(0, h - 1) \cap \{C_1 \cup \cdots \cup C_k\}.
\]