On a New Kato Class and Singular Solutions of a Nonlinear Elliptic Equation in Bounded Domains of $\mathbb{R}^n$*

HABIB MÁAGLI and MALEK ZRIBI
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 1060 Tunis, Tunisie (E-mail: habib.maagli@fst.rnu.tn)

(Received 5 August 2002; accepted 18 February 2004)

Abstract. Using a new form of the $3G$-Theorem for the Green function of a bounded domain $\Omega$ in $\mathbb{R}^n$, we introduce a new Kato class $K(\Omega)$ which contains properly the classical Kato class $K_n(\Omega)$. Next, we exploit the properties of this new class, to extend some results about the existence of positive singular solutions of nonlinear differential equations.


Key words: Green function, positive solution, Schauder fixed point theorem, singular nonlinear elliptic equation, singular solution

1. Introduction

Let $\Omega$ be a bounded $C^{1,1}$-domain in $\mathbb{R}^n (n \geqslant 3)$, and $G := G_\Omega$, be the Green function of the Laplacian in $\Omega$. In [13], Zhao have established interesting inequalities for the Green function $G$. In particular, he proved the existence of a positive constant $C$, such that for each $x, y, z$ in $\Omega$

\[ \frac{\delta(y)}{\delta(x)} G(x, y) \leqslant \frac{C}{|x - y|^{n-2}}, \quad (1.1) \]
\[ \frac{1}{C} H(x, y) \leqslant G(x, y) \leqslant CH(x, y), \quad (1.2) \]
\[ \frac{G(x, z)G(z, y)}{G(x, y)} \leqslant \left[ \frac{|x - y|^{n-2}}{|x - z|^{n-2}|y - z|^{n-2}} \right], \quad (1.3) \]

where

\[ H(x, y) := \frac{1}{|x - y|^{n-2}} \min \left( 1, \frac{\delta(x)\delta(y)}{|x - y|^2} \right) \]

* This paper has not been submitted elsewhere in identical or similar form, nor will it be during the first three months after its submission to Positivity.
and $\delta(x)$ denotes the Euclidean distance between $x$ and $\partial\Omega$.

The inequality (1.3), called $3G$-Theorem is often used in this form

$$
\frac{G(x, z)G(z, y)}{G(x, y)} \leq C \left( \frac{1}{|x-z|^{n-2}} + \frac{1}{|y-z|^{n-2}} \right). \tag{1.4}
$$

This $3G$-Theorem is useful for the study of functions belonging to the Kato class $K_n(\Omega)$ (see Definition 1 below), which is widely used in the study of some nonlinear differential equations (see for example [1], [10] and [12]). More properties pertaining to this class can be found in [1] and [3].

DEFINITION 1 (See [1] or [3]). A Borel measurable function $\phi$ in $\Omega$ belongs to the Kato class $K_n(\Omega)$ if $\phi$ satisfies the following condition

$$
\lim_{\alpha \to 0} \left( \sup_{x \in \Omega} \int_{\Omega \cap B(x, \alpha)} \frac{|\phi(y)|}{|x-y|^{n-2}} \, dy \right) = 0. \tag{1.5}
$$

In [6], Kalton and Verbitsky improve (1.4), in the following form

$$
\frac{G(x, z)G(z, y)}{G(x, y)} \leq C_0 \left[ \frac{\delta(z)}{\delta(x)} G(x, z) + \frac{\delta(z)}{\delta(y)} G(y, z) \right]. \tag{1.6}
$$

More precisely, they denoted by $N(x, y) = \frac{G(x, y)}{\delta(x)\delta(y)}$, the Naïm kernel and they proved in [6] (Lemma 7.1) that $\rho(x, y) = N(x, y)^{-1}$ is a quasi-metric on $\Omega$. Thus (1.6) holds.

This new form of the $3G$-Theorem allows us to introduce a new class of functions denoted by $K(\Omega)$ (see Definition 2 below), which contains properly the classical Kato class $K_n(\Omega)$ and which permits to generalize some results of [7], [10] and [12].

DEFINITION 2. A Borel measurable function $\phi$ in $\Omega$ belongs to the Kato class $K(\Omega)$ if $\phi$ satisfies the following condition

$$
\lim_{\alpha \to 0} \left( \sup_{x \in \Omega} \int_{\Omega \cap B(x, \alpha)} \frac{\delta(y)}{\delta(x)} G(x, y)|\phi(y)| \, dy \right) = 0. \tag{1.7}
$$

The first purpose of this paper is to study the properties of functions belonging to $K(\Omega)$, which we will doing in Section 2. In particular, we show for $1 \leq \lambda < 2$ that the function $x \to q(x) = \frac{1}{(\delta(x))^\lambda}$ is in $K(\Omega)$ but not in $K_n(\Omega)$. 