An Inequality from Moment Theory

Grahame Bennett

Abstract. We show how certain simple \( \ell^p \)-inequalities may be proved by “ignoring the \( p \).” An application to moment sequences is considered.

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1. Introduction

Our main result, Theorem 1.1 below, shows how certain simple \( \ell^p \)-inequalities may be proved by “ignoring the \( p \).” An application to moment sequences is considered in section 2.

Theorem 1.1. Suppose that \( a, b, c, x, y, z \) are positive numbers. Then the inequality

\[
 a^p + b^p + c^p \leq x^p + y^p + z^p
\]  

holds whenever \( p \geq 1 \), and reverses direction whenever \( p \leq 1 \), if and only if the following four conditions are satisfied:

\[
 a + b + c = x + y + z  
\]  

\[
 abc = xyz \]  

\[
 \max\{a, b, c\} \leq \max\{x, y, z\}  
\]  

\[
 \min\{a, b, c\} \leq \min\{x, y, z\} .  
\]

Inequality (1.1) is then strict, except when \( p = 0 \), or \( p = 1 \), or the sets \( \{a, b, c\} \) and \( \{x, y, z\} \) coincide.

Proof. (Necessity). If inequality (1.1) holds as stated, there must be equality when \( p = 1 \), so that (1.2) is guaranteed. To deduce (1.3)—(1.5), we first rephrase (1.1) in terms of \( L^p \)-means,

\[
 \left( \frac{a^p + b^p + c^p}{3} \right)^\frac{1}{p} \leq \left( \frac{x^p + y^p + z^p}{3} \right)^\frac{1}{p} .  
\]  

(1.6)
It is clear that (1.6) is valid whenever \( p \geq 1 \) or \( p < 0 \) and that the inequality reverses whenever \( 0 < p \leq 1 \). Making \( p \to \infty \) in (1.6), the means approach the corresponding maxima ([4], §2.3.4), forcing (1.4) to hold. (1.5) follows similarly by making \( p \to -\infty \). To prove (1.3), we make \( p \to 0 \) in (1.6), whereupon the \( L^p \)-means are replaced by the corresponding geometric means ([4], §2.3.3). When \( p \to 0^- \) we deduce that

\[
(abc)^{\frac{1}{3}} \leq (xyz)^{\frac{1}{3}},
\]

and, when \( p \to 0^+ \), that (1.7) is reversed.

(Sufficiency.) This, of course, is the point of the theorem: inequality (1.1) holds for all real \( p \) (in the directions indicated) if it holds at just four “points”, \( p = 1, \pm \infty \) and 0.

It will be convenient to assume, as we may, that the sets \( \{a, b, c\} \) and \( \{x, y, z\} \) are each arranged in descending order,

\[
a \geq b \geq c \quad \text{and} \quad x \geq y \geq z,
\]

in which case (1.4) and (1.5) are replaced by the simpler conditions

\[
a \leq x
\]

and

\[
c \leq z.
\]

We next invoke a result from the Theory of Majorization ([3], Theorem 10), according to which an inequality of the form

\[
\int_\alpha^\beta \phi(f(s))ds \leq \int_\alpha^\beta \phi(g(s))ds
\]

holds for all continuous convex functions, defined on \((0, \infty)\), if (and only if)

\[
\int_\alpha^\gamma f(s)ds \leq \int_\alpha^\gamma g(s)ds
\]

whenever \( \gamma \in (\alpha, \beta) \) and

\[
\int_\alpha^\beta f(s)ds = \int_\alpha^\beta g(s)ds.
\]

Here \( f \) and \( g \) are assumed to be non-negative integrable functions, both decreasing on \([\alpha, \beta]\).

We apply this result with \( \alpha = 0 \), \( \beta = b - y \),

\[
f(s) = \frac{1}{s + y} \quad \text{if} \quad 0 \leq s \leq b - y
\]

and

\[
g(s) = \begin{cases} 
\frac{1}{s + c} & \text{if} \quad 0 \leq s \leq z - c \\
\frac{1}{s + a + c - z} & \text{if} \quad z - c < s \leq b - y
\end{cases}
\]