New Version of the Daniell-Stone-Riesz Representation Theorem

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Abstract. The traditional representation theorems after Daniell-Stone and Riesz were in a kind of separate existence until Pollard-Topsøe 1975 and Topsøe 1976 were the first to put them under common roofs. In the same spirit the present article wants to obtain a unified representation theorem in the context of the author’s work in measure and integration. It is an inner theorem like the previous ones. The basis is the recent comprehensive inner Daniell-Stone theorem, so that in particular there are no a priori assumptions on the additive behaviour of the data.

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1. Introduction

The present article wants to put an adequate unified result on top of the collection of representation theorems of Daniell-Stone and Riesz type obtained in the context of the author’s work in measure and integration described in [12, 15]. We shall concentrate on the inner development which turned out to be more profound than the outer one. We recall that its basic concepts are the inner • premeasures \( \vartheta : \mathcal{G} \to [0, \infty] \) on a lattice \( \mathcal{G} \) with \( \emptyset \in \mathcal{G} \) in a nonvoid set \( X \) and their inner • extensions (• = \(*\sigma\tau \) with \(* = \) finite, \( \sigma = \) sequential, \( \tau = \) nonsequential), and that its basic devices are the inner • envelopes \( \vartheta : \mathcal{P}(X) \to [0, \infty] \) of the isotone set functions \( \vartheta : \mathcal{G} \to [0, \infty] \) with \( \vartheta(\emptyset) = 0 \). We shall often make free use of the concepts and results set up so far.

The basis is the inner Daniell-Stone representation procedure in [12] section 7. It assumed a function system \( E \subset [0, \infty]^X \) with \( 0 \in E \) and an isotone functional \( I : \)
\( E \to [0, \infty] \) with \( I(0) = 0 \), for the most part such that \( E \) is positive-homogeneous and a lattice under the pointwise max and min operations \( \lor \land \) which is Stonean: \( f \in E \Rightarrow f \land t, (f - t)^+ \in E \) for \( 0 < t < \infty \). But we emphasize that there are no a priori assumptions relative to the additive behaviour of \( E \) and \( I \). Then

\[
\text{Inn}(E) := \{ [f \geq t] : f \in E \text{ and } 0 < t < \infty \}
\]

is a lattice in \( X \) with \( \emptyset \in \text{Inn}(E) \). On \( \text{Inn}(E) \) one defines the inner sources of \( I \) to be the isotope set functions \( \varphi : \text{Inn}(E) \to [0, \infty] \) with \( \varphi(\emptyset) = 0 \) which represent \( I \) via the Choquet integral: \( I(f) = \int fd\varphi \) for all \( f \in E \). The inner sources fulfil \( I_*(\chi_X) \leq \varphi \leq I^*(\chi_X) \) on \( \text{Inn}(E) \), with the usual crude envelopes \( I_* \) and \( I^* \) of \( I \). For \( \bullet = \sigma \tau \) one defines \( I \) to be an inner \( \bullet \) preintegral iff there exist inner sources of \( I \) which are inner \( \bullet \) premeasures. The subsequent inner \( \bullet \) representation theorem \cite{12} 7.6 characterized these inner \( \bullet \) preintegrals and presented their basic properties. It is in terms of the inner \( \bullet \) envelopes \( I_* \) of \( I \) and their satellites. We recall that \( I \) Stonean means that \( I(f) = I(f \land t) + I((f - t)^+) \) for all \( f \in E \) and \( 0 < t < \infty \).

1.1 Inner \( \bullet \) Representation Theorem. Assume that \( E \subset [0, \infty]^X \) with \( 0 \in E \) is a positive-homogeneous Stonean lattice and \( I : E \to [0, \infty] \) with \( I(0) = 0 \) isotope. Then for \( \bullet = \sigma \tau \) one has the equivalences

\[
I \text{ is an inner } \bullet \text{ preintegral } \iff I \text{ is supermodular and Stonean and downward } \bullet \text{ continuous; and } I(v) \leq I(u) + I_*(v - u) \text{ for all } u \leq v \text{ in } E
\]

\[
I \text{ is supermodular and Stonean and downward } \bullet \text{ continuous at } 0; \text{ and } I(v) \leq I(u) + I^*_*(v - u) \text{ for all } u \leq v \text{ in } E.
\]

In this case there is a unique inner source of \( I \) which is an inner \( \bullet \) premeasure, and it is in fact \( \varphi = I^*(\chi_X)|\text{Inn}(E) \). This \( \varphi \) fulfills \( I_*(f) = \int fd\varphi \) for all \( f \in [0, \infty]^X \). Moreover the members of \( E_* \) are measurable \( \mathcal{E}(\varphi_*) \).

In the sequel we want to expand the above Daniell-Stone representation theorem so that it comprises the Riesz representation theorem in its recent comprehensive versions. For this purpose we assume besides \( E \) and \( I \) an additional lattice \( \mathcal{G} \) with \( \emptyset \in \mathcal{G} \) in \( X \). The aim is to characterize those isotope functionals \( I : E \to [0, \infty] \) with \( I(0) = 0 \) for which there exist inner \( \bullet \) premeasures \( \vartheta : \mathcal{G} \to [0, \infty] \) (or rather a unique one) which represent \( I : I(f) = \int fd\vartheta \) for all \( f \in E \), a formulation which makes sense for arbitrary \( \mathcal{G} \). In this context it is quite clear that one cannot expect substantial results without adequate connections between the two lattices \( \text{Inn}(E) \) and \( \mathcal{G} \). Justified by previous particular situations and by success, we shall impose the relations

\[
(\bullet) \quad \mathcal{G} \subset (\text{Inn}(E))_\bullet \quad \text{and} \quad \text{Inn}(E) \subset \mathcal{G} \sqcup \mathcal{G}_\bullet,
\]

with \( \sqcup \) the transporter; in the terms of \cite{12} section 4 this means that \( \mathcal{G} \) and \( \text{Inn}(E) \perp := \{ [f < t] : f \in E \text{ and } 0 < t < \infty \} \) form a \( \bullet \) complemental couple. Besides we need a simple formulation which expresses that the functional \( I \) is concentrated on \( \mathcal{G} \). There are three related candidates, which we present with some obvious implications.