Stochastic nonlinear Perron–Frobenius theorem

Igor V. Evstigneev · Sergey A. Pirogov

Received: 12 October 2007 / Accepted: 17 December 2008 / Published online: 8 January 2009
© Birkhäuser Verlag Basel/Switzerland 2009

Abstract We establish a stochastic nonlinear analogue of the Perron–Frobenius theorem on eigenvalues and eigenvectors of positive matrices. The result is formulated in terms of an automorphism \( T \) of a probability space and a random transformation \( D \) of the non-negative cone of an \( n \)-dimensional Euclidean space. Under assumptions of monotonicity and homogeneity of \( D \), we prove the existence of scalar and vector measurable functions \( \alpha > 0 \) and \( x > 0 \) satisfying the equation \( \alpha T x = D(x) \) almost surely. We apply the result obtained to the analysis of a class of random dynamical systems arising in mathematical economics and finance (von Neumann–Gale dynamical systems).

Keywords Random dynamical systems · Perron–Frobenius theory · Nonlinear cocycles · Stochastic equations · Random monotone mappings · Hilbert–Birkhoff metric · von Neumann–Gale dynamical systems

Mathematics Subject Classification (2000) Primary 37H10 · 37H99 · 37H15; Secondary 91B62 · 91B28

I. V. Evstigneev (✉)
Economics Department, University of Manchester, Oxford Road, Manchester M13 9PL, UK
e-mail: igor.evstigneev@manchester.ac.uk

S. A. Pirogov
Institute for Information Transmission Problems, Academy of Sciences of Russia, GSP-4, 101447 Moscow, Russia
e-mail: pirogov@mail.ru
1 Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space, and \(T : \Omega \to \Omega\) its automorphism, i.e., a one-to-one mapping such that \(T\) and \(T^{-1}\) are measurable and preserve the measure \(P\). For each \(\omega \in \Omega\), let \(D(\omega) = D(\omega, x)\) be a mapping of the set \(\mathbb{R}^n_+\) of non-negative \(n\)-dimensional vectors into itself, continuous and positively homogeneous (of degree one) in \(x\) and \(\mathcal{F}\)-measurable in \(\omega\). Define

\[
C(t, \omega) = D(T^{l-1} \omega)D(T^{l-2} \omega) \ldots D(\omega), \quad t = 1, 2, \ldots , \quad (1.1)
\]

where the product means the composition of maps, and \(C(0, \omega) = \text{Id}\) (the identity map). Then we have

\[
C(t, T^s \omega)C(s, \omega) = C(t+s, \omega), \quad t, s \geq 0, \quad (1.2)
\]

i.e., the mapping \(C(t, \omega)\) is a cocycle over the dynamical system \((\Omega, \mathcal{F}, P, T)\) (see, e.g., Arnold [1]). In what follows, it will be convenient to write \(C(t, \omega)x\) and \(D(\omega)x\) for the result of application of the corresponding map to the point \(x\).

For two vectors \(x = (x^1, \ldots, x^n)\) and \(y = (y^1, \ldots, y^n)\), we write \(x \leq y\) (resp. \(x < y\)) if \(x^i \leq y^i\) (resp. \(x^i < y^i\)) for all \(i\). The notation \(x < y\) means that \(x \leq y\) and \(x \neq y\). We write \(|x|\) for \(|x^1| + \cdots + |x^n|\). A mapping \(A : \mathbb{R}^n_+ \to \mathbb{R}^n_+\) is called monotone if \(Ax \leq Ay\) for any vectors \(x, y \in \mathbb{R}^n_+\) satisfying \(x \leq y\). It is called completely monotone if it preserves each of the relations \(x \leq y\), \(x < y\) and \(x < y\) between two vectors \(x, y \in \mathbb{R}^n_+\) (clearly, if \(A\) preserves the second relation, it preserves the first).

A mapping \(A\) is termed strictly monotone if the relation \(x < y\) implies \(A(x) < A(y)\).

We will assume that the mappings \(D(\omega) : \mathbb{R}^n_+ \to \mathbb{R}^n_+\) (\(\omega \in \Omega\)) are completely monotone and the cocycle \(C(t, \omega)\) satisfies the following condition.

(C) For almost all \(\omega \in \Omega\), there is a natural number \(l\) (depending on \(\omega\)) such that the mapping \(C(l, \omega)\) is strictly monotone.

The main result of this paper is as follows.

Theorem 1 (a) There exists a measurable vector function \(x(\omega) > 0\) and a measurable scalar function \(\alpha(\omega) > 0\) such that

\[
\alpha(\omega)x(T\omega) = D(\omega)x(\omega), \quad |x(\omega)| = 1 \text{ (a.s.)}. \quad (1.3)
\]

(b) The pair of functions \((\alpha(\cdot), x(\cdot))\) satisfying (1.3) is determined uniquely up to the equivalence with respect to the measure \(P\).

(c) If \(t \to \infty\), then

\[
\frac{C(t, T^{-l}\omega)a}{|C(t, T^{-l}\omega)a|} \to x(\omega) \text{ (a.s.)}, \quad (1.4)
\]

where convergence is uniform in \(a > 0\).

(d) Let \(\mathcal{F}_0\) and \(\mathcal{F}_1\) be sub-\(\sigma\)-algebras of \(\mathcal{F}\) such that the random maps \(D(T^{-1}\omega)x\), \(D(T^{-2}\omega)x\), \ldots are \(\mathcal{F}_0\)-measurable and the random maps \(D(\omega)x\), \(D(T^{-1}\omega)x\), \ldots