The dual of $C_{ps}(X)$

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Abstract. This is a study of the dual space of continuous linear functionals on the function space $C_{ps}(X)$ with a natural norm inherited from a larger Banach space. Here $ps$ denotes the pseudocompact-open topology on $C(X)$, the set of all real-valued continuous functions on a Tychonoff space $X$. The lattice structure and completeness of this dual space have been studied. Since this dual space is inherently related to a space of measures, the measure-theoretic characterization of this dual space has been studied extensively. Due to this characterization, a special kind of topological space, called $pz$-space, has been studied. Finally the separability of this dual space has been studied.


Keywords. Continuous linear functional, function space, Banach lattice, pseudocompact, pseudocompact-open topology, measure, zero set, $pz$-space, separability.

1. Introduction

The set $C(X)$ of all real-valued continuous functions as well as the set $C^*(X)$ of all bounded real-valued continuous functions on a Tychonoff space $X$ has a number of natural topologies. Two commonly used among them are the topology of uniform convergence $u$ and the compact-open topology $k$. While the topology of uniform convergence on $C(X)$ has been used for more than a century as the proper setting to study the uniform convergence of sequences of functions, the compact-open topology on $C(X)$ was shown in [14] to be the proper setting to study sequences of functions which converge uniformly on compact subsets. But the latter one also turned out to be a natural and interesting locally convex topology on $C(X)$ from the measure-theoretic viewpoint. In fact, continuous functions and Baire measures on Tychonoff spaces are linked by the process of integration. A number of natural locally convex topologies on spaces of continuous functions have been studied in order to clarify this relationship. For more information on these topologies, see [40].

The compact-open topology and the topology of uniform convergence on $C(X)$ (or on $C^*(X)$) are equal if and only if $X$ is compact. Because compactness
is such a strong condition, there is a considerable gap between these two topologies. The gap has been especially felt in the topological measure theory; consequently in the last five decades, there have been quite a few topologies introduced that lie between $k$ and $u$, such as the strict topology, the $\sigma$-compact-open topology, the topology of uniform convergence on $\sigma$-compact subsets and the topology of uniform convergence on bounded subsets. See for example, [3], [5], [8], [11], [17], [21], [22], [23], [31], [32].

The pseudocompact-open topology $ps$ is another natural and interesting locally convex topology on $C(X)$, from the viewpoint of both topology and measure theory. This topology has already been well-studied in [19], [20] and [18]. In particular, in these papers, the submetrizability, metrizability, completeness and countability properties of the pseudocompact-open topology on $C(X)$ have been studied. But since this topology is locally convex, it is quite natural to consider the corresponding dual space, more precisely to consider the space of continuous linear functionals on $C_{ps}(X)$, where $C_{ps}(X)$ denotes the space $C(X)$ equipped with the pseudocompact-open topology $ps$. In this paper, our goal is to study some basic properties of this dual space as well as its measure-theoretic characterization.

Similar to the compact-open topology $k$ on $C(X)$, the pseudocompact-open topology $ps$ on $C(X)$ is generated by the family of seminorms $\{p_A : A$ is a pseudocompact subset of $X\}$ where $p_A(f) = \sup\{|f(x)| : x \in A\}$ for $f \in C(X)$. Let $\mathcal{PS}(X)$ be the collection of all pseudocompact subsets of $X$. The basic open sets in $C_{ps}(X)$ look like $\langle f, A, \epsilon \rangle = \{g \in C(X) : |f(x) - g(x)| < \epsilon$ for all $x \in A\}$ where $f \in C(X)$, $\epsilon > 0$ and $A \in \mathcal{PS}(X)$. Since the closure of a pseudocompact subset of $X$ is again pseudocompact and $f(A) = \overline{f(A)}$ for all $f \in C(X)$ and $A \subseteq X$, we can always take closed pseudocompact subsets of $X$ in $\langle f, A, \epsilon \rangle$. Also since the family $\{(f, A, \epsilon) : f \in C(X), A \in \mathcal{PS}(X), \epsilon > 0\}$ is a base for the topology $ps$ on $C(X)$, the pseudocompact-open topology is actually the topology of uniform convergence on the pseudocompact subsets of $X$. Note that if $\mathcal{K}(X)$ is the collection of all compact subsets of $X$, then the family $\{(f, K, \epsilon) : f \in C(X), K \in \mathcal{K}(X), \epsilon > 0\}$ is a base for the compact-open topology $k$ on $C(X)$ and consequently this topology is weaker than the pseudocompact-open topology on $C(X)$. Like $C_{ps}(X)$, $C_{k}(X)$ denotes the space $C(X)$ equipped with the compact-open topology $k$.

In order to study the dual space of $C_{ps}(X)$ in an appropriate perspective, we need to bring the dual spaces of $C_{k}(X)$ and $C_{\infty}^*(X)$ also into focus. Here $C_{\infty}^*(X)$ is the well-known Banach space $C_{\infty}^*(X)$ equipped with the uniform or supremum norm $\|f\|_\infty$ where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ for $f \in C_{\infty}^*(X)$. Note that the topology on $C_{\infty}^*(X)$ induced by the norm $\|\cdot\|_\infty$ is actually the topology of uniform convergence, $u$, on $C_{\infty}^*(X)$. The dual spaces of $C_{k}(X)$, $C_{ps}(X)$ and $C_{\infty}^*(X)$, that is, the set of continuous linear functionals on $C_{k}(X)$, $C_{ps}(X)$ and $C_{\infty}^*(X)$, are denoted by $\Lambda_{k}(X)$, $\Lambda_{ps}(X)$ and $\Lambda_{\infty}(X)$ respectively. Note that since $C_{\infty}^*(X)$ is dense in $C_{j}(X)$, $(j = k, ps)$, $\Lambda_{j}(X)$ can also be considered as the set of all continuous linear functionals on $C_{j}^*(X)$, where $C_{j}^*(X)$ denotes the space $C_{j}^*(X)$ equipped with the topology $j$, $(j = k, ps)$. Also it is clear that $\Lambda_{k}(X) \subseteq \Lambda_{ps}(X)$.