Notes on the projective limit theorem of Kolmogorov

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Abstract The new systematization in measure and integration due to the author produced a version of the Kolmogorov projective limit theorem which is far more comprehensive than the previous ones. The present article is devoted to several consequences. In particular one obtains a topological version which applies to arbitrary Hausdorff spaces.

Keywords Inner premeasures · Sequential and nonsequential ones · Consistent families · Projective limits

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1 Introduction and preliminaries

The present article is part of the author’s new systematization in measure and integration, of which the latest introduction and account is in Ref. [13]. One of the fundamental points is the projective limit theorem of Kolmogorov [5]. Our new form of the theorem [9, 10] is for inner premeasures of mass one (=:prob), where this time we assume \( \bullet = \sigma \tau \). The assertion is the one-to-one correspondence between the appropriate inner premeasures on an infinite product space \( X = \prod_{t \in T} Y_t \) and the consistent families of such ones on the collection of all finite partial products \( Y_p = \prod_{t \in p} Y_t \). Our inner extension theorem with the incorporation of \( \bullet = \tau \) allows for the first time to overcome the
barrier of countably determined subsets of X in a natural manner, and thus to arrive at an adequate concept and treatment of stochastic processes [9–12].

The present paper wants to develop three consequences of our Kolmogorov type projective limit theorem. The first consequence in Sect. 3 is a topological version of the theorem which applies to all Hausdorff spaces. In contrast, the conventional theorems are restricted to Polish spaces or to Borel spaces, the Borel subspaces of Polish spaces, and in Rao [16] to Hausdorff spaces with countable base. The second consequence in Sect. 4 is concerned with the frequent form of the conventional projective limit theorem which assumes inner regularity not for all members of the relevant consistent families but restricted to the simplest ones which live on the factors Y_t. In our framework this is not a natural assumption, but we show that our theorem is apt to include the situation.

In all these cases we first present the results in our natural form for inner • prob premeasures on X. Then we specialize to the conventional forms for measures on σ algebras. As a rule these domains consist of countably determined subsets of X, and thus are much too small in case of an uncountable index set T. As a consequence it must be expected that the inner regularity structure of the resultant measures cannot be expressed.

The third consequence in Sect. 5 is devoted to the old method to obtain decent projective limit measures via compactification of X. For this extended matter we refer to Bogachev [2] vol. II, pp. 447–448. It is obvious that our new systematization renders the method obsolete. However, we cannot resist to demonstrate that it can lead to situations which are best described as compactification catastrophes: the basic space X is turned into an inner null set!

The present Sect. 1 continues with a few preliminaries, most of them previous results collected for convenience. Then Sect. 2 recalls our Kolmogorov type projective limit theorem, combined with a certain variant which will be useful in the sequel.

Preliminaries on set systems  Most of the basic terms are as defined in Refs. [6,8,14]. A nonvoid set system $\mathcal{S}$ on a nonvoid set $X$ is called a paving. We define $\mathcal{S}^* \subset \mathcal{S}^\sigma \subset \mathcal{S}^{\tau}$ to consist of the unions of the nonvoid finite/countable/arbitrary subsystems of $\mathcal{S}$, and $\mathcal{S}_* \subset \mathcal{S}_\sigma \subset \mathcal{S}_\tau$ to consist of the respective intersections. We also recall the shorthand notation • = ⋆στ. The first two remarks have obvious proofs.

Remark 1.1 Let $H : X \to Y$ be a map between nonvoid sets X and Y, and $\mathcal{B}$ be a paving in Y. Then

$$H^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) = \bigcup_{B \in \mathcal{B}} H^{-1}(B) \quad \text{and} \quad H^{-1}\left(\bigcap_{B \in \mathcal{B}} B\right) = \bigcap_{B \in \mathcal{B}} H^{-1}(B).$$

$$H^{-1}(\mathcal{B}^\bullet) = (H^{-1}(\mathcal{B}))^\bullet \quad \text{and} \quad H^{-1}(\mathcal{B}_\bullet) = (H^{-1}(\mathcal{B}))_\bullet.$$  

Remark 1.2 Let the $\mathcal{S}_l$ be pavings in $Y_l (l = 1, \ldots, n)$, and thus $\mathcal{S} := (\prod_{l=1}^n \mathcal{S}_l)^\bullet$ a paving in $Y := \prod_{l=1}^n Y_l$. If the $\mathcal{S}_l$ are lattices/rings/algebras then $\mathcal{S}$ is a lattice/ring/algebra as well.