Coderivatives of the generalized perturbation maps

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Abstract This paper is devoted to considering the coderivatives of the generalized perturbation maps in general Banach spaces. Under some mild conditions, the upper estimate of coderivatives of the generalized perturbation maps are obtained. Their exact calculus rules are obtained under some additional conditions. Furthermore, the generalized perturbation maps are shown to be differentiably regular under some strong conditions.

Keywords Set-valued maps · Normal cone · Coderivative · Strictly differentiable · Generalized perturbation map

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1 Introduction

Sensitivity analysis has a long history. It is not only theoretically interesting, but also practically important in optimization theory and in theory of variational inequality. By sensitivity, we mean the quantitative analysis, which is the study of derivatives of
perturbation maps. A number of interesting results of sensitivity analysis for perturbation maps have been obtained in \cite{4,7,8,10,11,22,24} among recent publications and also the references therein, where the derivative of a set-valued map (or multifunction, multimap) is defined from the classical geometric view: which regards the graph of the derivative at a point as the tangent to the graph of the set-valued map. Various graphical derivatives have been proposed by Aubin \cite{2}, Rockafellar \cite{25,26} and so on.

Another derivative-like construction for set-valued maps has been introduced by Mordukhovich \cite{13} using the normal cone to the graph. This object acting in the dual space is called the coderivative for the set-valued map under consideration at the point of its graph. A number of results of sensitivity analysis for perturbation maps by coderivatives have been obtained in \cite{6,9,19} and also the references therein.

Consider the generalized perturbation map $G$ from $Y \times Z$ to $X$ defined by

$$G(u, z) = \{x \in X \mid z \in F(x, u) + K(x, u)\}, \quad (1)$$

where $F$ and $K$ are set-valued maps from $X \times Y$ to $Z$.

Since many first-order conditions for optimization and variational inequality problems can be expressed in the form of generalized perturbation problems (1), there is much interest to discuss the sensitivity of (1). Levy and Rockafellar \cite{8} discussed the case when $F$ is a single-valued map and $K$ only depends on parameter $x$. In \cite{28}, Xue and Li obtained the sensitivity of (1) by the graphical derivatives. The solution maps of the “quasivariational inequalities” and “set-valued variational inequalities” problems which include a linear term as a part of their parametric dependence can also be expressed in the form (1), (see \cite{7,18,20,21}).

A more general form of the perturbation maps is

$$G(\lambda, u, v) = \{x \in C(\lambda) \mid v \in F(x, u) + K(\lambda, x)\}, \quad (2)$$

which is the solution maps of the parametric generalized equations: find $x \in C(\lambda)$ satisfying the following inclusion depending on a triple of parameters $(\lambda, u, v) \in Y \times U \times V$:

$$v \in F(x, u) + K(\lambda, x),$$

where $C : Y \rightrightarrows X$, $F : X \times U \rightrightarrows V$ and $K : Y \times X \rightrightarrows V$ are set-valued maps. This model includes some classes of problems and many important applications, some examples of this model can be found in \cite{10}. In finite dimensional spaces, Huy and Lee \cite{10} discussed the proto-differentiability of the solution map (2) and provided sufficient conditions for $G(\cdot)$ to be single-valued on a neighborhood of $(\lambda, u, v)$ and semidifferentiable at $(\lambda, u, v)$.

This paper is devoted to considering the sensitivity of the generalized perturbation maps of forms (1) and (2). In general Banach spaces, the Fréchet coderivative, normal coderivative and mixed coderivative of the generalized perturbation maps are discussed. Under some additional conditions, the generalized perturbation maps are shown to be differentially regular at the considered point. Since there are sums of two