Positivstellensätze for differentiable functions

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Abstract We present a canonical proof of both the strict and weak Positivstellensatz for rings of differentiable and smooth functions. Our construction is explicit, preserves definability in expansions of the real field, and it works in definably complete expansions of real closed fields as well as for real-valued functions on Banach spaces.

Keywords Positivstellensatz · Differentiable and smooth function · Definably complete structure · Banach space

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1 Introduction

Let $k$ be any nonnegative integer. Krivine’s result, cf. [5], known as Stengle’s Positivstellensatz, cf. [7], characterizes the nonnegative polynomials on the basic closed...
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semialgebraic set

\[ F := \{ f_1 \geq 0, \ldots, f_k \geq 0 \} \]

as follows: a polynomial \( g \) is nonnegative on \( F \) if and only if there are polynomials \( p_1, p_2 \) in the positive cone generated by the \( f_i \) and the sums of squares such that

\[ p_1g = p_2 + g^{2m} \]

for some \( m \geq 0 \).

Schmüdgen proved in [8] that if \( F \) is compact and \( g \) strictly positive on \( F \), then \( p_1 \) is not needed, and Putinar verified additional conditions for \( g \) belonging to the quadratic module generated by the \( f_i \). Using transcendental methods, Acquistapace, Andradas and Broglia proved in [1] strict and weak Positivstellensätze for smooth and differentiable functions on Euclidean spaces and differentiable functions in o-minimal expansions of real closed fields.

In the present note, we generalize and strengthen the results in [1]. In particular, we present an explicit finite method to construct the representation of the given positive function and we also deal with \( C^\infty \) functions which are hard to access with modeltheoretical methods.

Recall that an expansion of a real closed field \( R \) is definably complete, see [6], if every bounded definable subset of \( R \) has a supremum in \( R \). Notice that every structure expanding the real field is definably complete. So the structure consisting of all subsets of Euclidean spaces, and every o-minimal expansion of a real closed field is definably complete, cf. [3,4].

We work under the following general assumption.

Let \( 0 < d \in \mathbb{N} \) be fixed, let \( r \in \mathbb{N} \cup \{\infty\} \) be fixed, and let \( R \) be a real closed field.

1. If \( r = \infty \), we let \( \mathcal{R} \) be any definably complete expansion of \( R \) with a definable smooth exponential function.
2. If \( r \in \mathbb{N} \), we let \( \mathcal{R} \) be an arbitrary definably complete expansion of \( R \).

By definable, we always mean definable in \( \mathcal{R} \) with parameters from \( R \). A function is definable if its graph is definable.

For families of functions \( f_1,s, \ldots, f_k,s : M \to R, s \in S \), we let

\[ F_s := \cap_{i=1}^k \{ f_{i,s} \geq 0 \}. \]

We shall prove the following theorems.

**Theorem 1.1** (Definable Strict Positivstellensatz) Let \( M \) be a definable \( C^r \) manifold. Let \( S \subset R^m \) be a definable set. Let \( g_s, f_1,s, \ldots, f_k,s, s \in S \) be a definable family of \( C^r \) functions from \( M \) to \( R \) such that \( g_s > 0 \) on \( F_s \) and \( F_s \neq \emptyset \) for all \( s \). Then there are definable families of functions \( v_{0,s}, \ldots, v_{k,s} \in C^r (M, (0, \infty)) \) such that

\[ g_s = v_{0,s}^{2d} + \sum_{i=1}^k v_{i,s}^{2d} f_{i,s} \]

for all \( s \in S \).