On maximum and variational principles via image space analysis

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Abstract The analysis in the Image Space allows one to extend the applications of maximum and variational principles for constrained optimization. Such principles are embedded in a separation scheme, in the Image Space, which can be seen as a common root from which they are derived. In particular, Ekeland and Auchmuty Variational Principles are analysed.

Keywords Image space analysis · Separation and theorems of the alternative · Variational principles

Mathematics Subject Classification (2000) 49J40 · 65K10 · 90C30

1 Introduction

The main purpose of this paper consists in stressing the importance, in studying constrained extremum problems, of taking into consideration, among the various types of analyses, the Image Space Analysis (for short, ISA). The purpose will be pursued by considering some of the most known variational principles and showing, for each

This work was partially supported by the Grant NSC 99-2115-M-037-002-MY3.
of them, the possible advantages that might come from ISA. More than to survey the literature and to close problems, the paper aims at opening questions.

The idea of studying the properties of the image of the vector of the functions involved in a constrained extremum problem is an old one; a trace can be found even in the work of Caratheodory [5]. In the 1950s, R. Bellman, introducing his famous maximum principle [4], proposed to replace the unknowns by new ones, which run in the space of the images of the functions that define the given problem, the Image Space (IS). In the late 1970s and 1980s, some authors [10,14], independently of each other, explicitly carried out such a study in the field of Optimization.

The approach consists in introducing the IS and a new extremum problem, called image problem, which is equivalent to the given one, call it $P$, even though $P$ is not an extremum problem itself. The optimality condition for the image problem, and in turn for $P$, is equivalently expressed by the disjunction of two suitable subsets $\mathcal{K}$ and $\mathcal{H}$ of the IS associated with $P$: $\mathcal{K}$ is defined by the vector of the images of the functions involved in $P$, while $\mathcal{H}$ is a convex cone that depends on the type of constraints which define the feasible set of $P$. The disjunction between $\mathcal{K}$ and $\mathcal{H}$ is proved by showing that they lie in two disjoint level sets of a separating functional. When such a functional can be found linear, then we say that $\mathcal{K}$ and $\mathcal{H}$ admit linear separation. A suitable subclass of separating functionals is said to be regular, iff the separation obtained by means of the zero-level of a functional of such a subclass guarantees the disjunction between $\mathcal{K}$ and $\mathcal{H}$.

Exploiting separation arguments in the IS, several theoretical aspects can be developed, as duality, Lagrangian-type optimality conditions, regularity, and penalization [9,10].

The analysis in the IS must be viewed as a preliminary and auxiliary step for studying an optimization problem. Once a statement has been achieved in the IS, then an equivalent statement must be obtained in the given space. By means of the geometric interpretations arising from the ISA, it is possible to characterize, in a more handy form, suitable properties of the considered problem and to obtain generalizations that are almost unconceivable, by merely performing the analysis in the given space; in fact, once they have been stated, they appear far from the intuition applied to the given space.

Let us recall the main notations and definitions that will be used in the sequel.

$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$; a set $\mathcal{H} \subseteq \mathbb{R}^n$ is said to be a cone, with apex at the origin, iff $\lambda \mathcal{H} \subseteq \mathcal{H}$, with $\lambda \in \mathbb{R}_+$ \{0\}, and a convex cone iff, in addition, $\mathcal{H} + \mathcal{H} \subseteq \mathcal{H}$, where $\mathcal{H} + \mathcal{H} = \{h_1 + h_2 \in \mathbb{R}^n : h_1 \in \mathcal{H}, h_2 \in \mathcal{H}\}$; cone $\mathcal{H}^* := \{y \in \mathbb{R}^n : y = \lambda x, \; \lambda > 0, \; x \in \mathcal{H}\}$. The closure, the topological interior, the topological relative interior, and the convex hull of a set $\mathcal{H}$ are denoted by $cl \mathcal{H}$, $int \mathcal{H}$, $ri \mathcal{H}$, and $conv \mathcal{H}$, respectively. $\mathcal{H}^* := \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \; \forall x \in \mathcal{H}\}$ is the positive polar of $\mathcal{H}$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$. Let $a, b \in \mathbb{R}^m$ and $C \subseteq \mathbb{R}^m$ be a closed and convex cone; $a \geq_C b$ iff $a - b \in C$.

Let $X$ be a subset of a Banach space and $f : X \rightarrow \mathbb{R}^m$. $f$ is called $C$-function on the convex set $\mathcal{X} \subseteq X$, iff $\forall x_1, x_2 \in \mathcal{X}$, we have:

$$(1 - \alpha) f(x_1) + \alpha f(x_2) - f((1 - \alpha)x_1 + \alpha x_2)) \in C, \; \forall \alpha \in [0, 1].$$