Bismut Type Formulae for Diffusion Semigroups on Riemannian Manifolds

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Abstract Applying the stochastic calculus of variations on frame bundles along tangent processes, we derive Bismut type formulae for the derivatives of diffusion semigroups on Riemannian manifold in both variables. We also obtain the Bismut formulae expressed in terms of the Ricci and torsion tensors for the connection with torsion.

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Key words Bismut formulae · diffusion semigroups · heat kernel derivatives.

1. Introduction

Starting from the work of Bismut (cf. [2]) various formulae for derivatives (both in the first as well as in the second variable) of the heat kernel associated to elliptic operators on manifolds have been derived. These formulae are important since they exhibit smoothing properties of the corresponding semigroup and have other applications, namely to the study of small time asymptotics of the heat kernel. Neither the formulae nor the methods of proof are unique: while Bismut’s original work relied upon the use of Malliavin calculus and its integration by parts theorem, other

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approaches developed by Elworthy and Li are more elementary (cf. [9] and [10]). We refer also to [15, 16] for related formulae.

One can start from a general elliptic operator to which one associates a Riemannian structure (like in [9] and [10]) or start from the Riemannian structure itself and the corresponding operators. Following this last point of view, Bismut’s formula applies to the Laplace–Beltrami operator; the work of Driver ([7]) is an extension to Laplacians associated to connections with a skew-symmetric torsion (cf. also [11], where an integration by parts and a Weitzenbock formula is shown in this framework).

In the present work we consider the case of a general Riemannian connection and derive simple formulae expressed in terms of the Ricci tensor (that we explicitly compare with the Ricci for the Levi-Civita connection) for the derivatives of the heat kernel in both variables. We use integration by parts on the path space of the underlying manifold.

2. Bismut Formulae for Diffusion Semigroups

Let \((M, \langle \cdot, \cdot \rangle)\) be a connected and complete \(C^\infty\) Riemannian manifold of dimension \(d\) equipped with the Levi-Civita connection \(\nabla\). Let \(\Delta\) be the Laplace–Beltrami operator. We consider the following diffusion operator:

\[
L := \frac{1}{2} \Delta + Z,
\]

where \(Z\) is a \(C^1\)-vector field on \(M\).

Let \(\{A_j, j = 1, \cdots, d\}\) be the canonical horizontal vector fields relative to \(\nabla\). We consider the following Stratonovich SDEs on frame bundle \(O(M)\):

\[
\begin{cases}
    dr_\omega(t) = A_j(r_\omega(t))[\omega d\omega^j(t) + z^j(r_\omega(t))dt], \\
    r_\omega(0) = r_0 \in O(M),
\end{cases}
\]

where \(\{\omega(t)\}_{t \geq 0}\) is the standard \(d\)-dimensional Brownian motion on classical Wiener space, and \(z^j(r) := [r^{-1} Z(\pi(r))]^j\) is the scalarization of vector field \(Z, r \in O(M), \pi\) is the bundle projection from \(O(M)\) to \(M\). We use the standard Einstein’s convention for summations.

Set \(\gamma_\omega(t) := \pi(r_\omega(t))\). Then the generator of diffusion process \(\gamma_\omega(t)\) on \(M\) is given by \(L\). The diffusion process \(r_\omega(t)\) on \(O(M)\) is called the horizontal lift of \(\gamma_\omega(t)\) through the connection \(\nabla\). In this work, we assume that

the life time of \(\gamma_\omega(t, m_0)\) is infinite,

where \(m_0 = \pi(r_0)\). The diffusion semigroup associated to \(L\) is given by

\[
P_t f(m_0) := \mathbb{E}(f(\gamma_\omega(t, m_0))),
\]

where \(f \in C^\infty_0(M)\) has compact support.

Fixing \(T > 0\), we consider the path space \(\mathbb{P}_{m_0}(M)\) of the continuous paths from \([0, T]\) to \(M\) starting from \(m_0\). Let \(\mu\) be the Wiener measure on classical Wiener space \(\mathbb{P}_0(\mathbb{R}^d)\) and

\[
I^Z : \mathbb{P}_0(\mathbb{R}^d) \mapsto \mathbb{P}_{m_0}(M); \quad I^Z(\omega) = \gamma_\omega(\cdot, m_0).
\]