Anharmonic Oscillators in the Complex Plane, \(\mathcal{PT}\)-symmetry, and Real Eigenvalues

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Abstract For integers \(m \geq 3\) and \(1 \leq \ell \leq m - 1\), we study the eigenvalue problems
\[-u''(z) + [(-1)^\ell (iz)^m - P(iz)]u(z) = \lambda u(z)\]
with the boundary conditions that \(u(z)\) decays to zero as \(z\) tends to infinity along the rays \(\arg z = -\frac{\pi}{2} \pm \frac{(\ell + 1)\pi}{m+2}\) in the complex plane, where \(P\) is a polynomial of degree at most \(m - 1\). We provide asymptotic expansions of the eigenvalues \(\lambda_n\). Then we show that if the eigenvalue problem is \(\mathcal{PT}\)-symmetric, then the eigenvalues are all real and positive with at most finitely many exceptions. Moreover, we show that when \(\gcd(m, \ell) = 1\), the eigenvalue problem has infinitely many real eigenvalues if and only if one of its translations or itself is \(\mathcal{PT}\)-symmetric. Also, we will prove some other interesting direct and inverse spectral results.

Keywords Anharmonic oscillators · Asymptotics of the eigenvalues · \(\mathcal{PT}\)-symmetry

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1 Introduction

In this paper, we study Schrödinger eigenvalue problems with real and complex polynomial potentials in the complex plane under various decaying boundary conditions. We provide explicit asymptotic formulas relating the index \(n\) to a series of fractional powers of the eigenvalue \(\lambda_n\) (see Theorem 1.1). Also, we recover the polynomial potentials from asymptotic formula of the eigenvalues (see Theorem 1.4 and Corollary 2.3) and apply them to the so-called \(\mathcal{PT}\)-symmetric Hamiltonians (see Theorems 1.2 and 1.3).
For integers $m \geq 3$ and $1 \leq \ell \leq m - 1$, we consider the Schrödinger eigenvalue problem

$$(H_\ell u)(z) := \left[ -\frac{d^2}{dz^2} + (-1)^\ell (iz)^m - P(iz) \right] u(z) = \lambda u(z), \quad \text{for some } \lambda \in \mathbb{C},$$

(1.1)

with the boundary condition that

$$u(z) \to 0 \text{ as } z \to \infty \text{ along the two rays } \arg z = -\frac{\pi}{2} \pm \frac{(\ell + 1)\pi}{m + 2},$$

(1.2)

where $P$ is a polynomial of degree at most $m - 1$ of the form

$$P(z) = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z + a_m, \quad a_j \in \mathbb{C} \text{ for } 1 \leq j \leq m.$$