On Weak*-Convergence in $H^1_L(\mathbb{R}^d)$

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Abstract Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where $V$ is a nonnegative function, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. In this paper, we prove a version of the classical theorem of Jones and Journé on weak*-convergence in the Hardy space $H^1_L(\mathbb{R}^d)$.

Keywords Weak*-convergence · Schrödinger operator · Hardy space · VMO

Mathematics Subject Classifications (2010) 42B35 · 46E15

1 Introduction

A famous and classical result of Fefferman [7] states that the John-Nirenberg space $BMO(\mathbb{R}^d)$ is the dual of the Hardy space $H^1(\mathbb{R}^d)$. It is also well-known that $H^1(\mathbb{R}^d)$ is one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $VMO(\mathbb{R}^d)$ denote the closure of the space $C^\infty_c(\mathbb{R}^d)$ in $BMO(\mathbb{R}^d)$, where $C^\infty(\mathbb{R}^d)$ is the set of $C^\infty$-functions with compact support. Coifman and Weiss showed in [1] that $H^1(\mathbb{R}^d)$ is the dual space of $VMO(\mathbb{R}^d)$, which gives to $H^1(\mathbb{R}^d)$ a richer structure than $L^1(\mathbb{R}^d)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^d)$, but are bounded on $H^1(\mathbb{R}^d)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^d)$, which is useful in the application of Hardy spaces to compensated compactness (see [2]). More precisely, in [9], Jones and Journé proved the following.
Theorem J–J Suppose that \( \{ f_j \} \geq 1 \) is a bounded sequence in \( H^1(\mathbb{R}^d) \), and that \( f_j(x) \to f(x) \) for almost every \( x \in \mathbb{R}^d \). Then, \( f \in H^1(\mathbb{R}^d) \) and \( \{ f_j \} \geq 1 \) weak*-converges to \( f \), that is, for every \( \varphi \in VMO(\mathbb{R}^d) \), we have

\[
\lim_{j \to \infty} \int_{\mathbb{R}^d} f_j(x) \varphi(x) \, dx = \int_{\mathbb{R}^d} f(x) \varphi(x) \, dx.
\]

The aim of this paper is to prove an analogous version of the above theorem in the setting of function spaces associated with Schrödinger operators.

Let \( L = -\Delta + V \) be a Schrödinger differential operator on \( \mathbb{R}^d \), \( d \geq 3 \), where \( V \) is a nonnegative potential, \( V \neq 0 \), and belongs to the reverse Hölder class \( RH_{d/2} \). In the recent years, there is an increasing interest on the study of the problems of harmonic analysis associated with these operators, see for example [4–6, 10, 11, 13, 14]. In [6], Dziubański and Zienkiewicz considered the Hardy space \( H^1_L(\mathbb{R}^d) \) as the set of functions \( f \in L^1(\mathbb{R}^d) \) such that \( \| f \|_{H^1_L} := \| M_L f \|_{L^1} < \infty \), where \( M_L f(x) := \sup_{t > 0} |e^{-tL} f(x)| \). There, they characterized \( H^1_L(\mathbb{R}^d) \) in terms of atomic decomposition and in terms of the Riesz transforms associated with \( L \). Later, in [5], Dziubański et al. introduced a \( BMO \)-type space \( BMO_L(\mathbb{R}^d) \) associated with \( L \), and established the duality between \( H^1_L(\mathbb{R}^d) \) and \( BMO_L(\mathbb{R}^d) \). Recently, Deng et al. [4] introduced and developed new \( \dot{V}MO \)-type function spaces \( \dot{V}MO_A(\mathbb{R}^d) \) associated with some operators \( A \) which have a bounded holomorphic functional calculus on \( L^2(\mathbb{R}^d) \). When \( A = L \), their space \( \dot{V}MO_L(\mathbb{R}^d) \) is just the set of all functions \( f \) in \( BMO_L(\mathbb{R}^d) \) such that \( \gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0 \), where

\[
\begin{align*}
\gamma_1(f) &= \lim_{r \to 0} \left( \sup_{x \in \mathbb{R}^d, r \leq 1} \left( \frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-sL} f(y)|^2 \, dy \right)^{1/2} \right), \\
\gamma_2(f) &= \lim_{R \to \infty} \left( \sup_{x \in \mathbb{R}^d, r \geq R} \left( \frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-sL} f(y)|^2 \, dy \right)^{1/2} \right), \\
\gamma_3(f) &= \lim_{R \to \infty} \left( \sup_{B(x, t) \cap B(0, R) = \emptyset} \left( \frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-sL} f(y)|^2 \, dy \right)^{1/2} \right).
\end{align*}
\]

The authors in [4] further showed that \( H^1_L(\mathbb{R}^d) \) is in fact the dual of \( \dot{V}MO_L(\mathbb{R}^d) \), which allows us to study the weak*-convergence in \( H^1_L(\mathbb{R}^d) \). This is useful in the study of the Hardy estimates for commutators of singular integral operators related to \( L \), see for example Theorem 7.1 and Theorem 7.3 of [10].

Our main result is the following theorem.

Theorem 1.1 Suppose that \( \{ f_j \} \geq 1 \) is a bounded sequence in \( H^1_L(\mathbb{R}^d) \), and that \( f_j(x) \to f(x) \) for almost every \( x \in \mathbb{R}^d \). Then, \( f \in H^1_L(\mathbb{R}^d) \) and \( \{ f_j \} \geq 1 \) weak*-converges to \( f \), that is, for every \( \varphi \in VMO_L(\mathbb{R}^d) \), we have

\[
\lim_{j \to \infty} \int_{\mathbb{R}^d} f_j(x) \varphi(x) \, dx = \int_{\mathbb{R}^d} f(x) \varphi(x) \, dx.
\]

Throughout the whole paper, \( C \) denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In \( \mathbb{R}^d \), we denote by \( B = B(x, r) \) an open ball with center \( x \) and radius \( r > 0 \). For any measurable set \( E \), we denote by \( |E| \) its Lebesgue measure.